Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics I. *)

by

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A B S T R A C T

We give a theory of oscillatory integrals in infinitely many dimensions which extends the finite dimensional theory. In particular we extend the method of stationary phase, the theory of Lagrange immersions and the corresponding asymptotic expansions to the infinite dimensional case. A particular application of the theory to the Feynman path integrals defined in a previous paper by the authors yields asymptotic expansions to all orders of quantum mechanical quantities in powers of Planck's constant.

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Contents of Part I

1. Introduction

Oscillatory integrals in finitely many dimensions of the form

$$I(h) = \int_{R^d} e^{\frac{i}{h}\Phi(x)} g(x)dx \,, \tag{1.1}$$

where h is a parameter and Φ, g suitable smooth real functions, are a classical object of study. In particular one is interested in discussing the asymptotic behaviour of $I(h)$ when the parameter h goes to zero. Well known examples of simple integrals of the above form are the Fresnel integrals of the theory of wave diffraction and Airy's integrals of the theory of the rainbow. More generally such integrals arise quite naturally in the study of partial differential equations and in particular of those describing wave phenomena. The study of the asymptotic behaviour of integrals of the form (1.1) when $h \to 0$ is the subject of the well known classical method (Stokes, Kelvin,...) of the stationary phase and the related saddle point methods, see e.g. [1].

More recently a new vigorous investigation of oscillatory integrals of the form (1.1) has been initiated from two different points of view. One by Maslov, mainly in connection with the study of the classical limit of quantum mechanics, and the other by Hörmander, who developed the theory of Fourier integral operators as a powerful tool in the study of partial differential equations. For the work of Maslov see [2] and for the work of Hörmander see Ref. [3], and references therein, as well as Ref. [4], [5]. For the study of the case of degenerate critical points of the phase function φ in (1.1) it turns out to be useful to consider integrals of the more general form

$$I(h,y) = \int_{R^d} e^{\frac{i}{h}\Phi(x,y)} g(x,y)dx \ , \qquad (1.2)$$

where y is a new parameter in R^k . The theory of unfoldings of singularities applied to the discussion of the asymptotic behaviour of such integrals as $h \to 0$ has brought the study of finite dimensional oscillatory integrals and the corresponding stationary phase method to a high level of mathematical perfection. See in particular Arnold's work [6]. The reason why the study of integrals of the form (1.2), with the additional parameters $y \in R^K$, yields more information also on the original integral (1.1) is simple. One can namely control easily the asymptotic behaviour for $h \to 0$ of integrals of the form (1.1) by the standard classical method of stationary phase only when the phase function $\Phi(x)$ has only non degenerate stationary points. This case is generic in the sense that, by Morse theory, the set of functions $\Phi(x)$ with such property is open and dense in the set of all C^∞ functions and the complement has in a natural sense codimension one. Hence the situation where $\Phi(x)$ does have degenerate stationary points is unstable under perturbations arbitrary small in the C^∞ topology. If however Φ depends on additional parameters y then, by Thom's transversality theorem, see e.g. [9], there exists an open dense set of functions $y \to \Phi(\cdot,y)$ from R^k into $C^\infty(R^d)$ such that for each function in this set the function induced in the jet bundle over R^d intersects transversally the singular manifold in this jet bundle, hence the intersection is stable, a fact which makes it natural to study singularities of codimension $k \geq 1$ by studying k-dimensional families parameterized by $y \in R^k$.

The theory of oscillatory integrals in finitely many dimensions has received recently considerable attention also from another point of view. Thanks particularly to the work of Hörmander, see e.g. [3], and also [4], [5], a calculus of Fourier integral operators has been developed which greatly generalizes methods for existence and regularity for elliptic and pseudoelliptic operators to cover more general differential operators and, above all, provides constructive tools for the solutions of the corresponding equations. Integrals of the form (1.2) are naturally incorporated in Hörmander's theory of Fourier integral operators. A synthesis of the Maslov-Arnold line of work with Hörmander's one is contained in a recent paper of Duistermaat [5].

In the present paper we shall give a theory for the correspondent infinite dimensional case in which R^d is replaced by a real separable Hilbert space. The oscillatory integrals we treat are those which we introduced in [10] for the mathematical foundation of Feynman path integrals, and are natural generalizations of integrals of the form (1.1), and (1.2). Our study in the infinite dimensional situation uniformizes in particular the treatment of the finite dimensional cases. In all respects it generalizes the result on oscillatory integrals in finitely many dimensions, in particular the stationary phase method and all expansions in powers of h, to the infinite dimensional case.

When applied to the particular case of the Feynman path integrals of quantum mechanics as treated in [10], the corresponding asymptotic expansions in powers of h yield detailed results on the approach to the classical limit. Let us remark that these results justify one of the central arguments put forward formally by Dirac and Feynman for the formulation of quantum dynamics in terms of Feynman path integrals, namely that the classical limit should be a natural outcome of this formulation.

Independently of motivations concerning applications to the Feynman path integrals and the classical limit of quantum mechanics, we observe that our results lead in a natural way to a theory of Fourier integral operators in infinitely many dimensions and thus provide entirely new methods for the study of partial differential operators in infinitely many dimensions [11].

The present paper is part I of a series of two papers. We shall now briefly outline the content of part I.

In section 2 we develop the method of the stationary phase for oscillatory integral of the form

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{h}\Phi(x)} g(x)dx \ , \qquad (1.3)$$

where $\tilde{\int_{\mathcal{H}}}$ is the normalized integral on the real separable Hilbert space $\mathcal{H}$ defined in [10], and $\Phi(x)$ is of the form $\frac{1}{2}x^2 - V(x)$, where $V(x)$ is the Fourier transform of a complex measure such that $\Phi(x)$ has a single non degenerated stationary point in $\mathcal{H}$. We get in particular explicit asymptotic expansions in powers of h, to all orders in h, with control on the remainder.

In section 3 the oscillatory integrals (1.3) are studied in a more general situation where the phase function $\Phi(x)$ can have several stationary points. The method used is analogous to the one described above in connection with the usefulness of considering parametric integrals of the form (1.2) for the further study of integrals of the form (1.1). In particular we follow closely ideas of the theory of Hörmander [3] and of [5].

In section 4 we take up the study of the oscillatory integrals of the form (1.3) in the case where the phase function $\phi(x)$ can have stationary points which are degenerate. The line of attack is parallel to the one used for oscillatory integrals in finitely many dimensions, as developed in Hörmander [3],1) and Duistermaat [5]. In fact we show that this approach has a completely analogue extension to the infinite dimensional case we are considering. We formulate this extension by following very closely Ref. [5], and the reader is urged to read this reference, as well as [3],1), in parallel to our developments in order to fully appreciate the content of this section and the results obtained.

In section 5 we apply the methods of the previous sections to the study of the asymptotic approach to the classical limit from quantum mechanics. Let us first briefly mention some previous related work, which goes under the generic name of semiclassical approximations, see e.g. [12]. A subset of such studies belongs to the circle of ideas around the classical JKWB method (see e.g. [1],2), [13]) which is the version for Schrödinger's equation of the more general method of asymptotic expansions for differential equations. Well known classical applications are e.g. in the study of the relations between wave optics and geometrical optics. Related methods are used e.g. in [14]. The recent developments of Ref. [2] - [8] extend all such methods. For other methods used in the particular case of the approach to the classical limit from quantum mechanics see e.g. [15].[1)] The method we develop in the present paper for this problem is a mathematical version of the formal procedure used originally by Dirac and Feynman, see e.g. [20], [21] (more references are [1], [2] of [10]) and pursued later in [22], [23] . We also note that this beautiful heuristic idea of Dirac and Feynman has been a source of inspiration for the work of Pauli and

Choquard, [24], [25], as well as for Maslov's work [26].

Coming now to our work, we give a mathematical realization of the idea by using the theory of oscillatory integrals in infinitely many dimensions and their asymptotic expansions, as developed in sections 2-4, and applying it to the particular oscillatory integrals which occur in our formulation of the Feynman path solutions of Schrödinger's equation in Ref. [10]. In fact, the expansions of these Feynman path integrals in powers of Planck's constant $\hbar$ is reduced to a problem concerning finite dimensional oscillatory integrals and by this means we recover the asymptotic series in powers of $\hbar$ for the solution of Schrödinger's equation.

2. The method of the stationary phase in the analytic case.

Let $\mathcal{H}$ be a real separable Hilbert space with inner product $x \cdot y$. In [10] the normalized integral over $\mathcal{H}$

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} f(x)dx \tag{2.1}$$

was defined for the class $\mathcal{F}(\mathcal{H})$ of Fresnel integrable functions f, where $f \in \mathcal{F}(\mathcal{H})$ if and only if f is the Fouriertransform of a bounded complex measure on $\mathcal{H}$, i.e.

$$f(x) = \int_{\mathcal{H}} e^{ix\alpha} d\mu(\alpha) , \tag{2.2}$$

and the definition of (2.1) is simply

$$\int_{\mathcal{H}} e^{\frac{i}{2}x^2} f(x)dx = \tilde{\int_{\mathcal{H}}} e^{-\frac{i}{2}\alpha^2} d\mu(\alpha) . \tag{2.3}$$

In [10] it was proven that $\mathcal{F}(\mathcal{H})$ is a Banach function algebra in the natural norm $\|f\|_0 \equiv \|\mu\|$, where $\|\mu\|$ is the total variation of μ , and the normalized integral (2.1) is a bounded continuous normalized linear functional on $\mathcal{F}(\mathcal{H})$.
We shall consider now the Hilbert space $\mathcal{H}$ with the scaled inner product $\frac{1}{h}x \cdot y$, where h is a positive real number, and correspondingly we consider the normalized integral on $\mathcal{H}$ normalized with respect to this scaled inner product.
It follows from (2.2) and (2.3) that, if $f(x)$ is given by (2.2), then

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} f(x)dx = \int_{\mathcal{H}} e^{-\frac{ih}{2}\alpha^2} d\mu(\alpha) , \tag{2.4}$$

where the integral on the left hand side is normalized with respect to the inner product $\frac{1}{h}x \cdot y$.

We remark that $\mathcal{F}(\mathcal{H})$ is independent of h. Let now V ahd g be in $\mathcal{F}(\mathcal{H})$, then we have that $\exp(-\frac{i}{h}V)\cdot g$ is also in $\mathcal{F}(\mathcal{H})$, since $\mathcal{F}(\mathcal{H})$ is a Banach algebra. In this section we shall study the asymptotic behaviour as $h \to 0$ of integrals of the form

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx , \tag{2.5}$$

where $V(x)$ is a real function.

Since g and V are in $\mathcal{F}(\mathcal{H})$ we have

$$V(x) = \int_{\mathcal{H}} e^{i\alpha x} d\mu(\alpha) \tag{2.6}$$

and

$$g(x) = \int_{\mathcal{H}} e^{i\beta x} d\nu(\beta) , \tag{2.7}$$

where μ and ν are bounded complex measures.

In what follows we shall always assume that the first and second moments of the measure μ exist i.e.

$$\int_{\mathcal{H}} \alpha^2 \, d|\mu|(\alpha) < \infty , \tag{2.8}$$

where $|\mu|(\alpha)$ is the absolute value of μ and α^2 is the square of the norm of α. This implies that $V(x)$ is twice continuously Fréchet differentiable. Hence the Fréchet derivative $dV(x)$ exists and, since $\mathcal{H}$ is self dual, it may be identified with an element in $\mathcal{H}$. In fact we get from (2.6) that

$$dV(x) = i \int_{\mathcal{H}} \alpha e^{i\alpha x} d\mu(\alpha) , \tag{2.9}$$

so $x \to dV(x)$ is a continuous function from $\mathcal{H}$ into $\mathcal{H}$.
From the assumption (2.8) we get also that the function $x \to dV(x)$ is Fréchet differentiable and its Fréchet derivative $d^2V(x)$ at

the point x is a continuous linear function from $\mathcal{H}$ into $\mathcal{H}$ i.e. an operator on $\mathcal{H}$, in fact a bounded symmetric operator. From (2.9) we get

$$y \cdot d^2V(x)y = -\int_{\mathcal{H}} (\alpha \cdot y)^2 \, e^{i\alpha x} \, d\mu(\alpha) \; . \tag{2.10}$$

From (2.10) we have that $d^2V(x)$ is an integral with respect to the measure $e^{i\alpha x} d\mu(\alpha)$ of the one dimensional operator $-\alpha^2 \cdot P_\alpha$, where P_α is the orthogonal projection onto the subspace spanned by α. Hence

$$d^2V(x) = -\int \alpha^2 P_\alpha \, e^{i\alpha x} \, d\mu(\alpha) \; , \tag{2.11}$$

where the integral may be taken as a strong integral in the Banach space of bounded operators. It follows also from (2.10) that $d^2V(x)$ is of trace class and

$$\|d^2V(x)\|_1 \leq \int \alpha^2 \, d|\mu|(\alpha) \; , \tag{2.12}$$

where $\| \ \|_1$ is the trace norm. This follows from the following formula

$$\|d^2V(x)\|_1 = \sup \sum_n |y_n \cdot d^2V(x)y_n| \; , \tag{2.13}$$

where the supremum is taken over all complete orthonormal systems $\{y_n\}$. But by the triangle inequality

$$\sum_n |y_n \cdot d^2V(x)y_n| \leq \sum_n \int_{\mathcal{H}} (\alpha \cdot y_n)^2 d|\mu|(\alpha) \; ,$$

hence

$$\sum_n |y_n \cdot d^2V(x)y_n| \leq \int_{\mathcal{H}} \alpha^2 \, d\,|\mu|(\alpha) \; .$$

In fact we get in the same way that $x \to d^2V(x)$ is a continuous function from $\mathcal{H}$ into $L_1(\mathcal{H})$, where $L_1(\mathcal{H})$ is the Banach space of trace class operators.

In fact

$$\|d^2V(x) - d^2V(y)\|_1 \leq \int \alpha^2 |1 - e^{i(x-y)\cdot\alpha}| d|\mu(\alpha)| .$$

Lemma 2.1.

Let $V(x) = \int_{\mathcal{H}} e^{ix\alpha} d\mu(\alpha)$ be a real function such that $\int_{\mathcal{H}} \alpha^2 \, d|\mu|(\alpha) < 1$, then the equation

$$y = dV(y)$$

has a unique solution in $\mathcal{H}$.

Proof: From (2.9) and the triangle inequality we get

$$\|dV(x) - dV(y)\| \leq \int \|\alpha\| |1 - e^{i(x-y)\cdot\alpha}| d|\mu|(\alpha)$$

where $\|\alpha\|$ is the norm of α, so that

$$\|dV(x) - dV(y)\| \leq \|x-y\| \int \alpha^2 \left| \frac{1 - e^{i(x-y)\cdot\alpha}}{(x-y)\cdot\alpha} \right| d|\mu|(\alpha) .$$

Now $|\frac{1}{t}(1 - e^{it})|$ is a bounded function of t, in fact

$$|\tfrac{1}{t}(1 - e^{it})| = |\tfrac{1}{t}(e^{\frac{it}{2}} - e^{-\frac{it}{2}})| = 2\left|\frac{\sin\frac{t}{2}}{t}\right| \leq 1 .$$

Hence we have that

$$\|dV(x) - dV(y)\| \leq \|x-y\| \int \alpha^2 d|\mu|(\alpha) . \qquad (2.14)$$

By the assumption of the lemma and the contraction mapping principle the lemma is proven. $\square$

Lemma 2.2.

Let $V(x) = \int_{\mathcal{H}} e^{ix\alpha} d\mu(\alpha)$ where μ is a finite complex measure such that

$$\int_{\mathcal{H}} e^{\sqrt{2}|\alpha|} d|\mu|(\alpha) < 1$$

and such that $V(0) = 0$ and $dV(0) = 0$, and let $g(x) = \int_{\mathcal{H}} e^{ix\alpha} d\nu(\alpha)$, where ν is a bounded complex measure with $\int e^{\sqrt{2}|\beta|} d|\nu|(\beta) < \infty$, then

$$I(h) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx$$

is analytic in $\operatorname{Im} h < 0$, and C^∞ on the real line. Moreover

$$I(0) = |1 - d^2V(0)|^{-\frac{1}{2}} g(0) ,$$

where $|1 - d^2V(0)|$ is the determinant of the operator $1 - d^2V(0)$.

Proof: Since $e^{-\frac{i}{h}V} = \sum_{n=o}^{\infty} (\frac{-i}{h})^n \frac{1}{n!} V^n$, where the sum is norm convergent in $\mathcal{F}(\mathcal{H})$, we have by the properties of the normalized integral that

$$I(h) = \sum_{n=o}^{\infty} (\frac{-i}{h})^n \frac{1}{n!} \int \cdots \int e^{-\frac{i}{2}h(\sum_{j=1}^{n} \alpha_j + \beta)^2} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) \tag{2.15}$$

and outside a neighborhood of zero the series is uniformly convergent in $\operatorname{Im} h \leq 0$ and each term is obviously analytic there. From the assumptions on V we have that

$$\int d\mu(\alpha) = 0 \quad \text{and} \quad \int \alpha \, d\mu(\alpha) = 0 , \tag{2.16}$$

so that the $n-1$ first derivatives of

$$\int \cdots \int e^{-\frac{i}{2}h(\Sigma\alpha_j + \beta)^2} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) \tag{2.17}$$

with respect to h are zero at zero, hence each term in (2.15) is C^∞ on R . Consider now the function

$$q(t) = \frac{1}{(it)^n} [e^{it} - \sum_{s=o}^{n-1} \frac{(it)^s}{s!}] . \tag{2.18}$$

We verify easely that $q(t)$ is given by

$$q(t) = \frac{1}{t^n} \int\cdots\int_{t\geq t_1\geq\cdots\geq t_n\geq o} e^{it_n}\, dt_1\ldots dt_n \tag{2.19}$$

so that

$$|q(t)| \leq \frac{1}{n!} \ . \tag{2.20}$$

Now (2.15) may be written

$$I(h) = \sum_{n=o}^{\infty} \frac{(-i)^n}{n!} \int\cdots\int \left(\frac{-ih}{2}\left(\sum_{j=1}^{n} \alpha_j+\beta\right)^2\right)^{-n} \left[e^{-\frac{i}{2}h(\Sigma\alpha_j+\beta)^2} - \sum_{s=o}^{n-1} \frac{\left(-\frac{ih}{2}\left(\sum_{j=1}^{n}\alpha_j+\beta\right)^2\right)^s}{s!}\right]$$

$$\cdot\left(\frac{-i}{2}\left(\sum_{j=1}^{n} \alpha_j+\beta\right)^2\right)^n \prod_{j=1}^{n} d\mu(\alpha_j)\,d\nu(\beta) \ , \tag{2.21}$$

and by (2.20) we have that the sum of the absolute values of the terms is bounded by

$$\sum_{n=o}^{\infty} \frac{2^{-n}}{(n!)^2} \int\cdots\int\left(\left(\sum_{j=1}^{n}\alpha_j+\beta\right)^2\right)^n \prod_{j=1}^{n} d|\mu|(\alpha_j)\,d|\nu|(\beta)$$

$$\leq \sum_{n=o}^{\infty} \frac{2^{-n}}{(n!)^2} \int\cdots\int\left(\sum_{j=1}^{n}|\alpha_j|+|\beta|\right)^{2n} \prod_{j=1}^{n} d|\mu|(\alpha_j)\,d|\nu|(\beta) \tag{2.22}$$

$$= \sum_{n=o}^{\infty} \frac{2^{-n}}{(n!)^2} \sum_{k_o+k_1+\cdots+k_n=2n} \frac{(2n)!}{k_o!\cdots k_n!} \int\cdots\int \prod_{j=1}^{n} |\alpha_j|^{k_j} d|\mu|(\alpha_j)\,|\beta|^{k_o} d|\nu|(\beta).$$

Consider now the power series for $x > 0$:

$$\sum_{n=o}^{\infty} \frac{2^{-n}\,x^n}{(n!)^2} \sum_{k_o+k_1+\cdots+k_n=2n} \frac{(2n)!}{k_o!\cdots k_n!} \int\cdots\int \prod_{j=1}^{n} |\alpha_j|^{k_j} d|\mu|(\alpha_j)\,|\beta|^{k_o} d|\nu|(\beta). \tag{2.23}$$

By Stirling's formula for $n!$ we get that the power series (2.23) has the same radius of convergence as the power series

$$\sum_{n=o}^{\infty} 2^n x^n \sum_{k_o+\dots+k_n=2n} \frac{1}{k_o!\dots k_n!} \int\dots\int \prod_{j=1}^{n} |\alpha_j|^{k_j} d|\mu|(\alpha_j)|\beta|^{k_o} d|\nu|(\beta)$$

$$\leq \sum_{n=o}^{\infty} \sum_{k_o,\dots,k_n\geq o} \int\dots\int \prod_{j=1}^{n} \frac{(\sqrt{2x}|\alpha_j|)^{k_j}}{k_j!} d|\mu|(\alpha_j) \frac{(\sqrt{2x}|\beta|)^{k_o}}{k_o!} d|\nu|(\beta) \tag{2.24}$$

$$= \sum_{n=o}^{\infty} (\int e^{\sqrt{2x}|\alpha|} d|\mu|(\alpha))^n \cdot \int e^{\sqrt{2x}|\beta|} d|\nu|(\beta) ,$$

which is bounded for $x = 1$ iff

$$\int e^{\sqrt{2}|\alpha|} d|\mu|(\alpha) < 1 \tag{2.25}$$

$$\int e^{\sqrt{2}|\beta|} d|\nu|(\beta) < \infty . \tag{2.26}$$

Hence the series in (2.15) is uniformly and absolutely convergent, which proves that $I(h)$ is a continuous bounded function on R , since each term in (2.15) is C^∞. Consider now $I^{(n)}(h)$, the n-th derivative of $I(h)$. By taking the n-th derivative of each term in (2.15) we obtain by the corresponding estimate that $I^{(n)}(h)$ is the uniform limit of C^∞ functions, which gives us that $I(h)$ is C^∞ on R under the conditions (2.25) and (2.26). For the value of $I(h)$ at zero we get by the uniform convergence from (2.15) that

$$I(0) = \sum_{n=o}^{\infty} (\frac{-1}{2})^n (\frac{1}{n!})^2 \int\dots\int (\sum_{j=1}^{n} \alpha_j+\beta)^{2n} \prod_{j=1}^{n} d\mu(\alpha_j)d\nu(\beta) . \tag{2.27}$$

Since $\int d\mu(\alpha) = 0$ and $\int \alpha\, d\mu(\alpha) = 0$ we have that all terms in $(\sum_{j=1}^{n} \alpha_j+\beta)^{2n}$ which are not quadratic in each $\alpha_1,\dots,\alpha_n$ do not contribute, hence

$$I(0) = \left[\sum_{n=o}^{\infty} (\tfrac{-1}{2})^n \, (\tfrac{1}{n!})^2 \int\cdots\int \, (\sum_{j=1}^{n} \alpha_j)_2^{2n} \prod_{j=1}^{n} d\nu(\alpha_j)\right] g(0) \quad , \tag{2.28}$$

where $(\sum_{j=1}^{n} \alpha_j)_2^{2n}$ is the sum over all terms in the expansion of $(\sum_{j=1}^{n} \alpha_j)^{2n}$ which are exactly quadratic in each $\alpha_1, \ldots, \alpha_n$.

On the other hand (2.25) implies that

$$\int \alpha^2 \, d|\mu|(\alpha) < 1 \tag{2.29}$$

so that $\|d^2V(0)\|_1 < 1$. Hence the determinant

$$|1 - d^2V(0)|^{-\frac{1}{2}}$$

exists and has a convergent power series expansion in the operator $d^2V(0)$. Using now that

$$|1 - d^2V(0)^{-\frac{1}{2}} = \exp[-\tfrac{1}{2} \text{ tr } \ln(1-d^2V(0))] \tag{2.30}$$

and the fact that

$$x \cdot d^2V(0)x = - \int_{\mathcal{H}} (\alpha x)^2 \, e^{ix\alpha} \, d\mu(\alpha) \tag{2.31}$$

we get, by expanding (2.25) in powers of $d^2V(0)$, an expansion that converges for $\|d^2V(0)\|_1 < 1$ and this expansion is in fact the series given in (2.28). This proves the lemma. $\square$

We say that y is a critical point of a function $f(y)$ if $df(y) = 0$. We shall also say that a critical point y of f is regular if $d^2f(y)$ is bounded with bounded inverse as a linear operator on $\mathcal{H}$.

By lemma 2.1 we have that if $V(x) = \int e^{ix\alpha} \, d\mu(\alpha)$ with μ a bounded complex measure and

$$\int \alpha^2 \, d|\mu|(\alpha) < 1 \tag{2.32}$$

then there is a unique critical point of the exponent $\frac{1}{2}x^2 - V(x)$. The next lemma shows that this unique critical point may be approximated from a finite dimensional situation.

Lemma 2.3.

Let
$$V(x) = \int e^{ix\alpha}\, d\mu(\alpha)$$

where μ is a finite complex measure such that $|\mu|$ has a finite second moment $\int \alpha^2 d|\mu|(\alpha)$. Then $V(x)$ is twice differentiable with continuous $d^2V(x)$. Let $\|d^2V(x)\|$ be the norm of $d^2V(x)$ as a linear operator on $\mathcal{H}$. If $\|d^2V(x)\| \le k < 1$ for all $x \in \mathcal{H}$, we have that the equation

$$y = dV(y)$$

has a unique solution in $\mathcal{H}$. Moreover let P_n be a sequence of finite dimensional projections in $\mathcal{H}$ which converges strongly to 1. Let $V_n(x) = V(P_n x)$, and y_n the unique solution of

$$y_n = dV_n(y_n) .$$

Then $P_n y_n = y_n$ and y_n converges strongly to y as n tends to infinity.

Proof: It follows by dominated convergence that if $\int \alpha^2 d|\mu|(\alpha) < \infty$ then $V(x)$ is twice continuously differentiable. That $d^2V(x)$ is strongly continuous as a mapping $x \to d^2V(x)$ of $\mathcal{H}$ into $B(\mathcal{H})$ follows from the fact that the norm is bounded by the trace norm and we easely prove the estimate for the trace norm

$$\|d^2V(x) - d^2V(y)\|_1 \le \int \alpha^2 |1 - e^{i(x-y)\alpha}| \, d|\mu|(\alpha) ,$$

which goes to zero as $x \to y$ by dominated convergence. Since

$d^2V(x)$ is continuous in x we get by the mean value theorem that

$$dV(x) - dV(y) = d^2V(\lambda x + (1-\lambda y))(x-y)$$

for some λ, $0 < \lambda < 1$. By the assumption we therefore have

$$|dV(x) - dV(y)| \leq k|x-y|$$

so that $x \to dV(x)$ is contractive. Hence there is a unique solution of $y = dV(y)$. Now since $d^2V_n(x) = P_n d^2V(P_n x)P_n$ we have that $\|d^2V_n(x)\| < k$ and $d^2V_n(x)$ depends continuously on x. Therefore $y_n = dV_n(y_n)$ has a unique solution y_n, and since $dV_n(x) = P_n dV(P_n x)$ we have that $Py_n = y_n$.

Let now $T(x) = dV(x)$ and $T_n(x) = dV_n(x)$ then by the contraction mapping theorem we have for any $x_o \in \mathcal{H}$ that

$$y = \lim_{m\to\infty} T^m(x_o) \tag{2.33}$$

and

$$y_n = \lim_{m\to\infty} T_n{}^m(x_o) . \tag{2.34}$$

But
$$V_n(x) = \int e^{iP_n x\cdot\alpha} d\mu(\alpha) = \int e^{ixP_n\alpha} d\mu(\alpha)$$

so that

$$T_n(x) - T(x) = \int (i\alpha\, e^{ix\alpha} - iP_n\alpha\, e^{ixP_n\alpha}) d\mu(\alpha) .$$

Hence

$$|T_n(x) - T(x)| \leq \int |(\alpha - P_n\alpha)e^{ix\alpha} + P_n\alpha(e^{ix\alpha} - e^{ixP_n\alpha})| d|\mu|(\alpha)$$

$$\leq \int |(1-P_n)\alpha| d|\mu|(\alpha) + \int |\alpha|\cdot|e^{ix(1-P_n)\alpha} - 1| d|\mu|(\alpha).$$

Since

$$|e^{ix(1-P_n)\alpha} - 1| \leq |x|\ |(1-P_n)\alpha|\ \left|\frac{e^{ix(1-P_n)\alpha}-1}{x(1-P_n)\alpha}\right|$$

and $|\frac{1}{t}(e^{it}-1)| \leq 1$ we then get

$$|T_n(x)-T(x)| \leq \int (1+|x|\cdot|\alpha|)|(1-P_n)\alpha|\, d|\mu|(\alpha) ,$$

which goes to zero by dominated convergence and this convergence is uniform for x in any bounded set in $\mathcal{H}$. Now T_n as well as T map $\mathcal{H}$ into the ball of radius $\int|\alpha|\,d|\mu|(\alpha)$, hence $T_n{}^m(x_o)$ as well as $T^m(x_o)$ are all in this ball, and in this ball $T_n(x)$ converges to $T(x)$ uniformly in x. From this and (2.24) and (2.25) it follows that $y_n \to y$ strongly as n tends to infinity. This proves the lemma.

Unfortunately lemma 2.2 is only applicable to the asymptotic behavior for small h of

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx \tag{2.35}$$

when the critical point of the exponent $\frac{1}{2}x^2 - V(x)$ is at $x = 0$.

If, however, $\mathcal{H}$ is a finite dimensional real Hilbert space, we have by the classical theory of oscillating integrals the following lemma.

<u>Lemma 2.4</u>.
Let $\mathcal{H}$ be finite dimensional and let V and g be C^∞ with g in L_1, such that the exponent $\frac{1}{2}x^2 - V(x)$ has a unique regular critical point a, i.e. $a = dV(a)$ and $1-d^2V(a)$ is invertible, and is such that $\frac{1}{2}a^2 - V(a) = 0$. Then the function $I(h)$,

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx ,$$

is C^∞ on $\mathbb{R}$, and

$$I(0) = |1-d^2V(a)|^{-\frac{1}{2}} g(a) .$$

Proof: The proof of this classical lemma may be found for instance in Ref.[3],1) prop. 1.2.3. Note here that the normalized integral on a finite dimensional Hilbert space by Prop. 2.1 of Ref. [10] coincides in the case $g(x)$ is in L_1 with the usual Lebesgue integral up to a factor, so that in fact

$$I(h) = (2\pi ih)^{-\frac{1}{2}\dim \mathcal{H}} \int e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx \tag{2.36}$$

where dx in (2.36) is the Lebesgue measure normalized so that the unit cube has volume 1. □

Lemma 2.5.

Let $\mathcal{H}$ be a real separable Hilbert space, and let $V(x) = \int_{\mathcal{H}} e^{ix\alpha} d\mu(\alpha)$ and $g(x) = \int_{\mathcal{H}} e^{ix\alpha} d\nu(\alpha)$ with

$$\int_{\mathcal{H}} e^{\sqrt{2}|\alpha|} d|\mu|(\alpha) < 1 \quad \text{and} \quad \int_{\mathcal{H}} e^{\sqrt{2}|\alpha|} d|\nu|(\alpha) < \infty$$

and such that $V(a) = \frac{1}{2}a^2$ where a is the unique point such that $dV(a) = a$. Then

$$\sum_{n=k}^{\infty} \left(\frac{-1}{2}\right)^n \frac{1}{n!(n-k)!} \int \ldots \int \left(\sum_{j=1}^{n} \alpha_j + \beta\right)^{2(n-k)} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) = 0$$

for $k = 1,2,\ldots$.

Proof: Let $I(h)$ be given as in (2.36), then by a previous calculation we have for $h \neq 0$

$$I(h) = \sum_{n=0}^{\infty} \left(\frac{-i}{h}\right)^n \frac{1}{n!} \int \ldots \int e^{-\frac{i}{2}h(\sum_{j=1}^{n}\alpha_j+\beta)^2} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) . \tag{2.37}$$

The assumption of the lemma implies the inequality (2.32) and hence a unique point $a \in \mathcal{H}$ such that $dV(a) = a$. Now if $\mathcal{H}$ is finite dimensional the assumptions of the lemma imply that $V(x)$

and $g(x)$ **are** C^{∞} and that $g(x)$ is in L_1. We therefore get by lemma 2.4 that $I(h)$ is C^{∞}, so that $I(h)$ tends to a limit as $h \to 0$. From (2.37) we **have then**

$$\sum_{n=k}^{\infty} \left(\frac{-1}{2}\right)^n \frac{1}{n!(n-k)!} \int \cdots \int \left(\sum_{j=1}^{n} \alpha_j + \beta\right)^{2(n-k)} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) = 0 \,. \tag{2.38}$$

We note that by estimates of the type (2.22) the sum and the integrals in (2.38) converge absolutely. Consider now the left hand side of the equation (2.38) in the general infinite dimensional case, and let P_n be a sequence of finite dimensional projections such that P_n converges strongly to the identity. Let $V_n(x) = V(P_n x)$ and a_n the unique points such that $a_n = dV_n(a_n)$. Then

$$V_n^*(x) = V_n(x) - V_n(a_n) + \tfrac{1}{2} a_n^2 \tag{2.39}$$

satisfies the assumptions of this lemma and one has

$$V_n^*(x) = \int e^{ix\alpha} \, d\mu_n(\alpha) \;,$$

μ_n being the bounded complex measure defined by

$$\int_{\mathcal{H}} f(\alpha) d\mu_n(\alpha) = \int_{\mathcal{H}} f(P_n \alpha) d\mu(\alpha) - c_n f(0) \,, \tag{2.40}$$

where $c_n = V_n(a_n) - \frac{1}{2} a_n^2 = V(a_n) - \frac{1}{2} a_n^2$ since $P_n a_n = a_n$ by lemma 2.3. Let now $g_n(x) = g(P_n x)$. Then

$$g_n(x) = \int e^{\,ix\alpha} \, d\nu_n(\alpha)$$

with

$$\int f(\alpha) d\nu_n(\alpha) = \int f(P_n \alpha) d\nu(\alpha) \tag{2.41}$$

also satisfies the assumptions of this lemma. Hence by what we have already observed in the finite dimensional situation we have that

$$\sum_{m=k}^{\infty}\left(\frac{-1}{2}\right)^m \frac{1}{m!(m-k)!}\int_{(P_n\mathscr{H})^m}\cdots\int \sum_{j=1}^{m}\alpha_j+\beta)^{2(m-k)}\prod_{j=1}^{m}d\mu_n(\alpha_j)d\nu_n(\beta) = 0, \quad (2.42)$$

on the other hand, by (2.40) and (2.41), this is equal to

$$\sum_{m=k}^{\infty}\left(\frac{-1}{2}\right)^m \frac{1}{m!(m-k)!}\int_{\mathscr{H}^m}\cdots\int (P_n(\sum_{j=1}^{m}\alpha_j+\beta))^{2(m-k)}\prod_{j=1}^{m}(d\mu(\alpha_j)-c_n\delta(\alpha_j))d\nu(\beta). \quad (2.43)$$

Now $c_n = V(a_n)-\frac{1}{2}a_n^2$ converges to zero since $V(x)$ is continuous and a_n converges to a by lemma 2.3, so that $c_n \to V(a)-\frac{1}{2}a^2$, which is zero by assumption. Moreover by estimates of the type (2.22) we have that the sum and the integrals in (2.43) are absolutely convergent and uniformly so in n. In fact if $|c_n| \leq c$ for all n we have that each integrand in (2.43) is bounded by

$$\left(\tfrac{1}{2}\right)^m \frac{1}{m!(m-k)!}\left(\sum_{j=1}^{m}|\alpha_j|+|\beta|\right)^{2(m-k)}\prod_{j=1}^{m}(d|\mu|(\alpha_j)+c\delta(\alpha_j))d|\nu|(\beta)$$

and by estimates of the type (2.22) we have that

$$\sum_{m=k}^{\infty}\left(\tfrac{1}{2}\right)^m \frac{1}{m!(m-k)!}\int\cdots\int\left(\sum_{j=1}^{m}|\alpha_j|+|\beta|\right)^{2(m-k)}\prod_{j=1}^{m}(d|\mu|(\alpha_j)+c\delta(\alpha_j))\,d|\nu|(\beta)$$

is finite. So by Lebesgue's theorem on dominated convergence we have that (2.43) converges to

$$\sum_{m=k}^{m}\left(\frac{-1}{2}\right)^m \frac{1}{m!(m-k)!}\int\cdots\int(\Sigma\alpha_j+\beta)^{2(m-k)}\prod_{j=1}^{m}d\mu(\alpha_j)d\nu(\beta) \quad (2.44)$$

and since (2.43) was zero for all n we have proved that (2.44) is also zero. This then proves the lemma. $\square$

We shall now see that we have the following theorem.

Theorem 2.1.

Let $V(x) = \int_{\mathcal{H}} e^{ix\alpha}\, d\mu(\alpha)$ where μ is a finite complex measure on $\mathcal{H}$ such that $\int_{\mathcal{H}} e^{\sqrt{2}|\alpha|}\, d|\mu|(\alpha) < 1$ and $g(x) = \int_{\mathcal{H}} e^{ix\alpha}\, d\nu(\alpha)$ with $\int_{\mathcal{H}} e^{\sqrt{2}|\beta|}\, d|\nu|(\beta) < \infty$. Then there is a unique point $a \in \mathcal{H}$ such that $dV(a) = a$. Let

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2}\, e^{-\frac{i}{h}V(x)}\, g(x)\, dx \;.$$

Then $I(h)$ is analytic in $\operatorname{Im} h < 0$, and

$$e^{-\frac{i}{h}(\frac{1}{2}a^2 - V(a))}\, I(h) = I^*(h)$$

is a C^∞ function of h on the real line. Moreover its value at zero is given by

$$I^*(0) = |1 - d^2V(a)|^{-\frac{1}{2}}\, g(a)$$

where $|1 - d^2V(a)|$ is the Fredholm determinant of the operator $1 - d^2V(a)$.

Proof: The analyticity in $\operatorname{Im} h < 0$ follows in the same way as in lemma 2.2. Now $I^*(h)$ is $I(h)$ computed with $V^*(x) = V(x) + \frac{1}{2}a^2 - V(a)$ instead of $V(x)$, so we may therefore just as well assume that $V(a) = \frac{1}{2}a^2$. Now we have as in (2.15)

$$I(h) = \sum_{n=0}^{\infty} \left(\frac{-i}{h}\right)^n \frac{1}{n!} \int \cdots \int e^{-\frac{i}{h}\left(\sum_{j=1}^{n}\alpha_j + \beta\right)^2} \prod_{j=1}^{n} d\mu(\alpha_j)\, d\nu(\beta) \;. \tag{2.45}$$

Since we assume $V(a) = \frac{1}{2}a^2$ we have by lemma 2.5 that

$$I(h) = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int \cdots \int \frac{1}{h^n}\left[e^{-\frac{i}{2}h\left(\sum_{j=1}^{n}\alpha_j+\beta\right)^2} - \sum_{s=0}^{n-1}\left(\frac{-ih}{2}\right)^s \frac{1}{s!}\left(\sum_{j=1}^{n}\alpha_j+\beta\right)^{2s}\right] \prod_{j=1}^{n} d\mu(\alpha_j)\, d\nu(\beta) \;. \tag{2.46}$$

Each term in (2.46) is obviously a C^{∞} function of h. Now (2.21) is the same expression as (2.46) and by estimating in the same way as for (2.21) we get that

$$I_m(h) = \sum_{n=o}^{m} (-i)^n \frac{1}{n!} \int\cdots\int \frac{1}{h^n} \Big[e^{-\frac{i}{2}h(\sum_{j=1}^{n}\alpha_j+\beta)^2} - \sum_{s=o}^{n-1} (\frac{-ih}{2})^s \frac{1}{s!} (\sum_{j=1}^{n} \alpha_j+\beta)^{2s} \Big] \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta) \tag{2.47}$$

converges as $m \to \infty$ uniformly on compacts together with all its derivatives. This then proves that $I^*(h)$ is C^{∞}. It follows from this that, if $V(a) = \frac{1}{2} a^2$, then $I(0)$ is given by

$$I(0) = \sum_{m=o}^{\infty} (\frac{-1}{2})^m (\frac{1}{m!})^2 \int\cdots\int (\Sigma\alpha_j+\beta)^{2m} \prod_{j=1}^{m} d\mu(\alpha_j) d\nu(\beta) . \tag{2.48}$$

However, this is the formula (2.44) with $k = 0$ and it follows in the same way as for $k \geq 1$ that (2.48) is the limit of

$$\sum_{m=o}^{\infty} (\frac{-1}{2})^m (\frac{1}{m!})^2 \int\cdots\int (P_n(\sum_{j=1}^{m} \alpha_j+\beta))^{2m} \prod_{j=1}^{m} (d\mu(\alpha_j) - c_n\delta(\alpha_j)) d\nu(\beta) \tag{2.49}$$

as $n \to \infty$, where P_n is a sequence of finite dimensional projections such that P_n converges strongly to 1. But (2.49) is, by what we have already seen, equal to $I(h,V_n^*,g_n)$ taken at $h = 0$, where $V_n^*(x) = V(P_n x) - c_n$ and $g_n(x) = g(P_n x)$, with $c_n = V_n(a_n) - \frac{1}{2} a_n^2$, a_n being the unique point such that $a_n = dV(a_n)$. Now by the fact that V_n^* and g_n only depend on a finite number of dimensions we have that the normalized integral $I(h,V_n^*,g_n)$ reduces to a finite dimensional integral by prop 2.2 of Ref. [10]. Hence by lemma 2.4 we get

$$I(0,V_n^*,g_n) = |1 - d^2V_n^*(a_n)|^{-\frac{1}{2}} g_n(a_n) . \tag{2.50}$$

We have, on the other hand, the trace norm estimate

$$\|d^2V_n^*(x) - d^2V(x)\| \leq \int |(1-P_n)\alpha|^2 \, d\,|\mu|(\alpha) + |c_n| \int \alpha^2 d|\mu|(\alpha) , \tag{2.51}$$

which goes to zero by dominated convergence, and

$$\|d^2V(x) - d^2V(y)\| \leq \int |\alpha|^2 \, |1 - e^{\,i\alpha(x-y)}| \; d|\mu|(\alpha) , \tag{2.52}$$

which also goes to zero as $x \to y$ by dominated convergence, Hence $d^2V_n^*(a_n) \to d^2V(a)$ in trace norm and since $g_n(a_n) = g(a_n) \to g(a)$, $g(x)$ being continuous, we get, by expanding (2.50) in powers of $d^2V_n^*(a_n)$, that (2.50) goes to

$$|1 - d^2V(a)|^{-\frac{1}{2}} \, g(a) \tag{2.53}$$

as $n \to \infty$. On the other hand (2.50) is equal to (2.49), which we have already seen converges to (2.48) as $n \to \infty$. This proves the theorem. □

<u>Corollary 2.1</u>.

Let $V(x)$ and $g(x)$ satisfy the assumptions of theorem 2.1, and let $b \in \mathcal{H}$ and

$$i(h) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} \, e^{-\frac{i}{h}(V(x)+b\cdot x)} \, g(x)dx .$$

Then $I(h)$ is analytic in $\operatorname{Im} h < 0$ and

$$I^*(h) = e^{-\frac{i}{h}(\frac{1}{2}a^2 - V(a) - ba)} I(h)$$

is a C^∞ function of h on R, where a is the unique point such that

$$dV(a) + b = a .$$

Moreover the value of $I^*(h)$ at $h = 0$ is given by

$$I^*(0) = |1 - d^2V(a)|^{-\frac{1}{2}} g(a) .$$

Proof: By the translation invariance of the normalized integral (Proposition 2.3 of Ref. [10]), we have that

$$I(h) = e^{\frac{i}{2h}b^2} \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x+b)} g(x+b)dx . \qquad (2.54)$$

Now $V(x+b)$ and $g(x+b)$ satisfy the assumptions of theorem 2.1, and hence the corollary follows from theorem 2.1, since $dV(y+b) = y$ has the solution $y = a-b$. This proves the corollary. $\square$

<u>Corollary 2.2</u>.

Let $V(x)$ and $g(x)$ be as in theorem 2.1 and let a be the unique point such that $dV(a) = a$. Let us also assume that $V(a) = \frac{1}{2}a^2$. Let P_n be a sequence of finite dimensional projections in $\mathcal{H}$ such that P_n converges strongly to 1. Let $V_n^*(x) = V(P_nx) - c_n$ with $c_n = V(a_n) - \frac{1}{2}a_n^2$, where a_n is given by $dV(P_na_n) = a_n$, and let $g_n(x) = g(P_nx)$. If

$$I(h;V,g) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx$$

and

$$I(h;V_n^*,g_n) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V_n^*(x)} g_n(x)dx$$

then

$$I(h;V_n^*,g_n) = \tilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V_n^*(x)} g_n(x)dx$$

and

$I(h;V_n^*,g_n)$ converges, uniformly on compacts together with all its derivatives with respect to h, to $I(h;V,g)$.

Proof: From (2.47) we have that

$$I_m(h;V,g)=\sum_{k=0}^{m}(-i)^k\frac{1}{k!}\int\cdots\int\frac{1}{h^k}\Big[e^{-\frac{i}{2}h(\sum_{j=1}^{k}\alpha_j+\beta)^2}-\sum_{s=0}^{k-1}\left(\frac{-ih}{2}\right)^s\frac{1}{s!}\Big(\sum_{j=1}^{k}\alpha_j+\beta\Big)^{2s}\Big]\prod_{j=1}^{k}d\mu(\alpha_j)\,d\nu(\beta) \tag{2.55}$$

converges uniformly on compacts with all its derivatives to $I(h;V,g)$ as $m\to\infty$. On the other hand we have by the same type of estimates as for (2.21) that

$$I_m(h,V_n^*,g_n)=\sum_{k=0}^{m}(-i)^k\frac{1}{k!}\int\cdots\int\frac{1}{h^k}\Big[e^{-\frac{i}{2}h|P_n(\sum_{j=1}^{k}\alpha_j+\beta)|^2}-\sum_{s=0}^{k-1}\left(\frac{-ih}{2}\right)^s\frac{1}{s!}\Big|P_n\Big(\sum_{j=1}^{k}\alpha_j+\beta\Big)\Big|^{2s}\Big]\prod_{j=1}^{k}d\mu(\alpha_j)\,d\nu(\beta) \tag{2.56}$$

converges uniformly on compacts with all its derivatives to (2.55) as $n\to\infty$ uniformly in m. **Moreover, again by** the same type of estimates as in (2.21), we get that $I_m(h,V_n^*,g_n)$ converges to $I(h,V_n^*,g_n)$ uniformly on compacts with all its derivatives as $m\to\infty$, uniformly in n. Combining these two convergences we have the corollary. □

<u>Corollary 2.3.</u>

Let $V(x)=\int_{\mathcal{H}}e^{ix\alpha}\,d\mu(\alpha)$ where μ is a finite complex measure on $\mathcal{H}$ such that, for some $\lambda>0$, $\frac{1}{\lambda^2}\int_{\mathcal{H}}e^{\sqrt{2}\lambda|\alpha|}\,d|\mu|(\alpha)<1$ and $g(x)=\int_{\mathcal{H}}e^{ix\alpha}\,d\nu(\alpha)$ with $\int_{\mathcal{H}}e^{\sqrt{2}\lambda|\beta|}\,d|\nu|(\beta)<\infty$. Then the conclusions of theorem 2.1 hold.

Proof: It follows immediately from the definition of the normalized integral that

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx$$

$$= \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}\lambda^2 x^2} e^{-\frac{i}{h}V(\lambda x)} g(\lambda x)dx ,$$

where, in accordance with our notations, the first integral is normalized with respect to the form $\frac{1}{h}x^2$ and the second with respect to the form $\frac{\lambda^2}{h}x^2$. So we therefore get that

$$I(\lambda^2 h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}\frac{1}{\lambda^2}V(\lambda x)} g(\lambda x)dx .$$

On the other hand we have that

$$\frac{1}{\lambda^2}V(\lambda x) = \frac{1}{\lambda^2}\int e^{ix\cdot\lambda\alpha}\, d\mu(\alpha) \text{ and } g(\lambda x) = \int e^{ix\cdot\lambda\beta}\, d\nu(\beta) ,$$

so by the assumption of the corollary we may apply theorem 2.1 to the function $I(\lambda^2 h)$. This proves the corollary. $\square$

<u>Corollary 2.4.</u>

Let $V(x)$ and $g(x)$ be as in theorem 2.1 and let a be the unique point such that $dV(a) = a$, and let also $V(a) = \frac{1}{2}a^2$. Then

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx$$

have the two indentical asymptotic expansions at zero

$$I(h) = \sum_{m=o}^{\infty} h^m \left(\frac{-i}{2}\right)^m \sum_{m=o}^{\infty} \left(\frac{-1}{2}\right)^n \frac{1}{n!(m+n)!}\int\cdots\int \left(\sum_{j=1}^{n}\alpha_j+\beta\right)^{2(m+n)} \prod_{j=1}^{n} d\mu(\alpha_j)d\nu(\beta)$$

$$= \sum_{m=o}^{\infty} h^m \left(\frac{-i}{2}\right)^m \sum_{n=o}^{\infty} \left(\frac{-1}{2}\right)^n \frac{1}{n!(m+n)!}\int\cdots\int \left(\sum_{j=1}^{n}\alpha_j+\beta\right)_2^{2(m+n)}$$

$$\prod_{j=1}^{n} e^{ia\alpha_j}\, d\mu(\alpha_j)\, e^{ia\beta}\, d\nu(\beta) ,$$

where $(\sum_{j=1}^{n}\alpha_j+\beta)_2^{2(m+n)}$ is the sum of all terms in the expansion of $(\sum_{j=1}^{n}\alpha_j+\beta)^{2(m+n)}$ which are at least quadratic in each α_j, $j = 1,\ldots,n$. Moreover we have the estimate

$$|I(h)-\sum_{m=o}^{N}h^m(\tfrac{-i}{2})^m\sum_{n=o}^{\infty}(\tfrac{-1}{2})^n\frac{1}{n!(m+n)!}\int\cdots\int(\sum_{j=1}^{n}\alpha_j+\beta)^{2(m+n)}\prod_{j=1}^{n}d\mu(\alpha_j)d\nu(\beta)|$$

$$\leq |h|^{N+1}(N+1)!\,(1-\int e^{\sqrt{2}|\alpha|}\,d|\mu|(\alpha))^{-N-2}\int e^{\sqrt{2}|\beta|}\,d|\nu|(\beta)\,.$$

Proof: That $I(h)$ has an asymptotic expansion follows from the fact that $I(h)$ is a C^∞ function, and the first form of its asymptotic expansion follows from (2.55) and the fact that $I_m(h;V,g)$ converges uniformly on compacts with all its derivatives to $I(h)$ as $m\to\infty$. Now if the critical point a is at $a = 0$, the two expansions are obviously identical. Suppose now $a \neq 0$, then let us first assume that $\mathcal{H}$ is finite dimensional i.e. $\mathcal{H} = R^l$ for some l, and let us suppose that, in addition to the conditions of theorem 2.1, $g \in \mathcal{S}(R^l)$. We then know that the normalized integral is equal to an ordinary integral

$$\begin{aligned} I(h) &= \widetilde{\int_{R^l}} e^{\frac{i}{2h}x^2}\, e^{-\frac{i}{h}V(x)}g(x)dx \\ &= (2\pi ih)^{-\frac{l}{2}}\int e^{\frac{i}{2h}x^2}\, e^{-\frac{i}{h}V(x)}\, g(x)dx\,. \end{aligned} \tag{2.57}$$

Let now $a \in R^l$ be the critical point so that $dV(a) = a$. We then have

$$I(h) = (2\pi ih)^{-l/2}\int_{R^l} e^{\frac{i}{2h}x^2}\, e^{-\frac{i}{h}(V(x+a)-x\cdot a-V(a))}g(x+a)dx, \tag{2.58}$$

since $V(a) = \frac{1}{2}a^2$. Expanding now $e^{-\frac{i}{h}(V(x+a)-x\cdot a-V(a))}$ in powers of $U(x) = V(x+a) - x\cdot a - V(a)$ and using that $U(0) = 0$ and $dU(0) = 0$ and the fact that

$$V(x+a) = \int e^{ix\cdot\alpha} e^{ia\cdot\alpha} d\mu(\alpha) \tag{2.59}$$

we get, in the same way as in the proof of lemma 2.2, that $I(h)$ has the following asymptotic expansion

$$I(h) = \sum_{m=o}^{\infty} h^m (\tfrac{-i}{2})^m \sum_{n=o}^{\infty} (\tfrac{-1}{2})^n \frac{1}{n!(m+n)!} \int\cdots\int \left(\sum_{j=1}^{n} \alpha_j + \beta\right)_2^{2(m+n)} \prod_{j=o}^{n} e^{ia\alpha_j} d\mu(\alpha_j)\, e^{ia\beta} d\nu(\beta) \ . \tag{2.60}$$

Hence we have proved the identity of the two expansions if $\mathcal{H}$ is finite dimensional and $g \in \mathcal{S}$. So if $\mathcal{H}$ is finite dimensional, we have actually proved the identities

$$\sum_{n=o}^{\infty} (\tfrac{-1}{2})^n \frac{1}{n!(m+n)!} \int\cdots\int \left(\sum_{j=1}^{n} \alpha_j + \beta\right)^{2(m+n)} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta)$$

$$= \sum_{n=o}^{\infty} (\tfrac{-1}{2})^n \frac{1}{n!(m+n)!} \int\cdots\int \left(\sum_{j=1}^{n} \alpha_j + \beta\right)_2^{2(m+n)} \prod_{j=1}^{n} e^{ia\alpha_j} d\mu(\alpha_j)\, e^{ia\beta} d\nu(\beta) \ , \tag{2.61}$$

if $\int e^{\sqrt{2}|\alpha|} d|\mu|(\alpha) < 1$ and $\int e^{\sqrt{2}|\beta|} d|\nu|(\beta) < \infty$ and $\nu(\beta) \in \mathcal{S}(R^1)$. Since $\mathcal{S}$ is weakly dense in the set of measures and the left hand side of (2.61) is finite for $\int e^{\sqrt{2}|\beta|} d|\nu|(\beta) < \infty$, we have that the right hand side is also finite and equal to the left hand side. Hence we have proved the identity of the two expansions for $\mathcal{H}$ finite dimensional. The identity of the two expansions for arbitrary separable $\mathcal{H}$ then follows from corollary 2.2. For the estimate we get from (2.21) that

$$|I(h) - \sum_{m=o}^{N} h^m (\tfrac{-i}{2})^m \sum_{n=o}^{\infty} (\tfrac{-1}{2})^n \frac{1}{n!(m+n)!} \int\cdots\int (\sum_{j=1}^{n} \alpha_j + \beta)^{2(m+n)} \prod_{j=1}^{n} d\mu(\alpha_j) d\nu(\beta)|$$

$$\leq \sum_{n=o}^{\infty} \frac{|h|^{N+1}}{n!} \int\cdots\int |(\tfrac{-ih}{2}(\sum_{j=1}^{n} \alpha_j+\beta)^2)^{-(n+N+1)} \Big[e^{-\frac{i}{2}h(\Sigma\alpha_j+\beta)^2} - \sum_{s=o}^{n+N} \frac{1}{s!} (\tfrac{-ih}{2}(\sum_{j=1}^{n}\alpha_j+\beta)^2)^s \Big]$$

$$\cdot (\tfrac{-i}{2}(\sum_{j=1}^{n} \alpha_j+\beta)^2)^{n+N+1} | \prod_{j=1}^{n} d|\mu|(\alpha_j) d|\nu|(\beta) \qquad (2.61)$$

$$\leq |h|^{N+1} \sum_{n=o}^{\infty} \frac{2^{-(n+N+1)}}{n!(n+N+1)!} \int\cdots\int \sum_{j=1}^{n} (|\alpha_j|+|\beta|)^{2(n+N+1)} \prod_{j=1}^{n} d|\mu|(\alpha_j) d|\nu|(\beta)$$

$$\leq |h|^{N+1} \sum_{n=o}^{\infty} 2^{-(n+N+1)} \frac{(2n+2N+1)!}{[(n+N+1)!]^2} \frac{(n+N+1)!}{n!}$$

$$\cdot \int\cdots\int \sum_{k_o+\cdots+k_n=2(n+N+1)} \frac{1}{k_o!\cdots k_n!} \prod_{j=1}^{n} |\alpha_j|^{k_j} d|\mu|(\alpha_j) |\beta|^{k_o} d|\nu|(\beta)$$

$$\leq |h|^{N+1} \sum_{n=o}^{\infty} 2^{n+N+1} \frac{(n+N+1)!}{n!} \int\cdots\int \sum_{k_o+\cdots+k_n=2(n+N+1)} \frac{1}{k_o!\cdots k_n!} \prod_{j=1}^{n} |\alpha_j|^{k_j} d|\mu|(\alpha_j) \, |\beta|^{k_o} d|\nu|(\beta|$$

$$= |h|^{N+1} \sum_{n=o}^{\infty} \frac{(n+N+1)!}{n!} \int\cdots\int \sum_{k_o+\cdots+k_n=2(n+N+1)} \frac{1}{k_o!\cdots k_n!} \prod_{j=1}^{n} |\sqrt{2}\alpha_j|^{k_j} d|\mu|(\alpha_j) \, |\sqrt{2}\beta|^{k_o} d|\nu|(\beta)$$

$$= |h|^{N+1} \sum_{n=o}^{\infty} \frac{(n+N+1)!}{n!} (\int e^{\sqrt{2}|\alpha|} d|\mu|(\alpha))^n \cdot \int e^{\sqrt{2}|\beta|} d|\nu|(\beta)$$

$$= |h|^{N+1} \cdot (N+1)! \; (1 - \int e^{\sqrt{2}|\alpha|} d|\mu|(\alpha))^{-N-2} \int e^{\sqrt{2}|\beta|} d|\nu|(\beta) \; .$$

This proves the corollary. $\square$

Corollary 2.5.

Let $V(x)$ and $g(x)$ be as in theorem 2.1, then $I(h)$ has the following asymptotic expansion

$$I(h) = \sum_{m=o}^{\infty} h^m (\tfrac{i}{2})^m \sum_{n=o}^{\infty} (\tfrac{1}{2})^n \frac{1}{n!(m+n)!} \left[(\sum_{j=1}^{n} \nabla_{x_j} + \nabla_y)_2^{2(m+n)} V(x_1)\dots V(x_n) g(y) \right],$$

where the value of [ ] is to be taken at $x_i = y = a$ and where a is the critical point, i.e. $\nabla V(a) = a$, and $V(a) = \frac{1}{2} a^2$.

$(\sum_j \nabla_{x_j} + \nabla_y)_2^{2(m+n)}$ is the sum of all terms in the expansion of $(\sum_j \nabla_{x_j} + \nabla_y)^{2(m+n)}$ which are of at least second degree with respect to each ∇_{x_j}, $j = 1, \dots, n$.

Proof: This is just the second version of the asymptotic expansion in the previous corollary. $\square$

3. The method of the stationary phase in the general case.

In the previous section we developed the method of the stationary phase for the case where the phase function

$$V(x) = \int_{\mathcal{H}} e^{i\alpha x} \, d\mu(\alpha) \tag{3.1}$$

is gentle and small. We shall say that $V(x)$ given by (3.1) is gentle if there is a $\lambda > 0$ such that

$$\|\mu\| < \lambda^2 \quad \text{and} \quad \int_{\mathcal{H}} e^{\sqrt{2}\lambda|\alpha|} \, d|\mu|(\alpha) < \infty , \tag{3.2}$$

and we shall say that $V(x)$ is small and gentle if

$$\frac{1}{\lambda^2} \int_{\mathcal{H}} e^{\sqrt{2}\lambda|\alpha|} \, d|\mu|(\alpha) < 1 \tag{3.3}$$

for some $\lambda > 0$. With this denominations, we have that theorem 2.1 and corollary 2.3 give the asymptotic expansion of

$$I(h) = \tilde{\int} e^{\frac{i}{2h}x^2} \, e^{-\frac{i}{h}V(x)} g(x) dx \tag{3.4}$$

for $V(x)$ small and gentle. We have seen that $V(x)$ small and gentle implies that the total phase function $\frac{1}{2}x^2 - V(x)$ has one and only one critical or stationary point a, $d(\frac{1}{2}x^2 - V(x))_{x=a} = 0$, and from corollary 2.5 we have that all the terms in the asymptotic expansion of $I(h)$ at $h = 0$ are given by the derivatives of $V(x)$ and $g(x)$ at the point a only.

We know that in the finite dimensional case, if we have several critical or stationary points, the asymptotic expansion of $I(h)$ is just the sum over all the stationary points of the corresponding expansion for each critical point. In the finite dimensional case this is namely immediately seen by writing $g(x)$ as a sum of functions each one with support containing only one critical point.

In our case, however, which is the infinite dimensional case, this can not be done because we do not know whether $I(h)$ has an asymptotic expansion at all, if

$$g(x) = \int e^{ix\beta} \, d\nu(\beta) \tag{3.5}$$

does not satisfy the condition

$$\int e^{\sqrt{2}\lambda|\beta|} \, d|\nu|(\beta) < \infty . \tag{3.6}$$

But if g satisfies (3.6) we have that, for $z = x + iy$, with x and y in $\mathcal{H}$,

$$g(z) = \int_{\mathcal{H}} e^{iz\cdot\beta} \, d\nu(\beta) \tag{3.7}$$

is analytic in $|y| < \sqrt{2}\lambda$, hence the support of $g(x)$ must be all of $\mathcal{H}$.

However, in this section we shall see how to overcome this problem.

So let now $V(x)$ be gentle i.e. there is a $\lambda > 0$ such that (3.2) holds and let $g(x)$ satisfy (3.6). Let P_n be orthogonal projections on $\mathcal{H}$ with finite dimensional ranges such that P_n converge strongly to the identity. By Lebesgue lemma on dominated convergence we then have that

$$\frac{1}{\lambda^2} \int_{\mathcal{H}} e^{\sqrt{2}\lambda|\alpha - P_n\alpha|} \, d|\mu|(\alpha) \to \frac{1}{\lambda^2} \int d|\mu|(\alpha) < 1 . \tag{3.8}$$

Hence there is some projection P with finite dimensional range such that

$$\frac{1}{\lambda^2} \int_{\mathcal{H}} e^{\sqrt{2}\lambda|(1-P)\alpha|} \, d|\mu|(\alpha) < 1 . \tag{3.9}$$

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = (1-P)\mathcal{H} \oplus P\mathcal{H}$, and we shall use the notation $x = (y,z)$ for $x = y \oplus z$. Since $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is isomorphic as a metric space, and therefore as a measure space, with $\mathcal{H}_1 \times \mathcal{H}_2$ we have that $d\mu(\alpha)$ may be considered as a measure $d\mu(\beta,\gamma)$ on the product space $\mathcal{H}_1 \times \mathcal{H}_2$, $\alpha = (\beta,\gamma)$. With the notation $V(x) = V(y,z)$ we then have

$$V(y,z) = \int_{\mathcal{H}_1 \times \mathcal{H}_2} e^{i\beta y} \cdot e^{i\gamma z} \, d\mu(\beta,\gamma) \; . \tag{3.10}$$

Let now $\mu_z(\beta)$ be the measure on $\mathcal{H}_1$ given by

$$\int_{\mathcal{H}_1} f(\beta) d\mu_z(\beta) = \int_{\mathcal{H}_1 \times \mathcal{H}_2} f(\beta) \, e^{i\gamma z} \, d\mu(\beta,\gamma) \; . \tag{3.11}$$

We then have

$$V(y,z) = \int_{\mathcal{H}_1} e^{i\beta y} \, d\mu_z(\beta) \; . \tag{3.12}$$

Moreover by the Minkowski inequality

$$\frac{1}{\lambda^2} \int_{\mathcal{H}_1} e^{\sqrt{2}\lambda|\beta|} \, d|\mu_z|(\beta) \le \frac{1}{\lambda^2} \int_{\mathcal{H}_1} e^{\sqrt{2}\lambda|\beta|} \, d|\mu|(\beta,\gamma) \; ,$$

so from (3.9) we have

$$\frac{1}{\lambda^2} \int_{\mathcal{H}_1} e^{\sqrt{2}\lambda|\beta|} \, d|\mu_z|(\beta) < 1 \; . \tag{3.13}$$

Hence we see that, for any fixed $z \in \mathcal{H}_2$, $V(y,z)$ is gentle and small as a function of $y \in \mathcal{H}_1$. From (3.13) it follows that

$$\int_{\mathcal{H}_1} \beta^2 \, d|\mu_z|(\beta) < 1 \tag{3.14}$$

so by lemma 2.1 we have that the equation

$$d_1 V(y,z) = y \; , \tag{3.15}$$

where $d_1 V(y,z)$ is the derivative of $V(y,z)$ with respect of y , has a unique solution $b(z)$:

$$d_1V(b(z),z) = b(z) . \tag{3.16}$$

We shall see that $b(z)$ is a smooth function from $\mathcal{H}_2$ into $\mathcal{H}_1$. By taking the derivative of (3.16) we get

$$d_1{}^2V(b(z),z)db(z) + d_1d_2V(b(z),z) = db(z) . \tag{3.17}$$

By (2.12) we have

$$\|d_1{}^2V(y,z)\|_1 \le \int_{\mathcal{H}_1} \beta^2 \, d|\mu_z|(\beta)$$

and by the Minkowski inequality we therefore have from (3.9) the following uniform estimate for the trace-norm

$$\|d_1{}^2V(y,z)\|_1 \le \int_{\mathcal{H}_1\times\mathcal{H}_2} \beta^2 \, d|\mu|(\beta,\gamma) < 1 .$$

Hence $1-d_1{}^2V(y,z)$ has a uniformly bounded inverse and from (3.17)

$$db(z) = (1-d_1{}^2V(b(z),z))^{-1} \, d_1d_2V(b(z),z) . \tag{3.18}$$

This then proves that $db(z)$ is uniformly bounded and continuous in z, so that $z \to b(z)$ is a smooth mapping. In fact it follows from the assumptions on V that $d_1V(y,z)$ is analytic in $|\mathrm{Im}\, y|^2 + |\mathrm{Im}\, z|^2 < 2\lambda^2$ and since $b(z)$ is a regular solution of

$$d_1V(b(z),z) = b(z)$$

we therefore get that $b(z)$ is real analytic from $\mathcal{H}_2$ into $\mathcal{H}_1$. We state this result in the following lemma.

<u>Lemma 3.1</u>. If $V(x)$ is gentle, i.e. satisfies (3.2), then there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with finite dimensional $\mathcal{H}_2$ such that, with $V(x) = V(y,z)$ for $x = y \oplus z$, $V(y,z)$ is, as a

function of y, gentle and small uniformly in z, i.e.

$$\frac{1}{\lambda^2} \int_{\mathcal{H}_1} e^{\sqrt{2}\lambda|\beta|} \, d|\mu_z|(\beta) \leq \frac{1}{\lambda^2} \int_{\mathcal{H}_1 \times \mathcal{H}_2} e^{\sqrt{2}\lambda|\beta|} \, d|\mu|(\beta,\gamma) < 1 .$$

Moreover the equation

$$d_1 V(y,z) = y$$

has a unique solution $y = b(z)$ for all $z \in \mathcal{H}_2$ and the mapping $z \to b(z)$ is real analytic from $\mathcal{H}_2$ into $\mathcal{H}_1$. □

Let us now consider the equation

$$d_2 V(b(z),z) = z \tag{3.19}$$

where $d_2 V(y,z)$ is the partial derivative of $V(y,z)$ with respect to z. Since $d_2 V(y,z)$ is analytic in y and z, we have that $d_2 V(b(z),z)$ is an analytic function on the finite dimensional space $\mathcal{H}_2$, so that (3.19) has at most a discrete set $\{z_i\}$ of solutions. Hence we have proved the following lemma.

Lemma 3.2.

If $V(x)$ is gentle, i.e. satisfies (3.2), then the equation

$$dV(x) = x$$

has at most a discrete set S of solutions, i.e. S has no limit points in $\mathcal{H}$. □

Let us now consider the Fresnel integral

$$I(h) = \widetilde{\int} e^{\frac{i}{2h} x^2} e^{-\frac{i}{h} V(x)} g(x) dx .$$

By Proposition 2.4 of ref. [10] (The Fubini theorem for Fresnel integrals) we have that, with the splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of lemma 3.1:

$$I(h) = \tilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2h}z^2} \Big(\tilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}y^2} e^{-\frac{i}{h}V(y,z)} g(y,z)dy\Big)dz \ . \qquad (3.20)$$

Now by lemma 3.1 $V(y,z)$ is, as a function of y, gentle and small and $d_1V(y,z) = y$ has a unique solution. We may therefore apply theorem 2.1 and its corollary 2.3 to the inner integral

$$I_1(h,z) = \tilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}y^2} e^{-\frac{i}{h}V(y,z)} g(y,z)dy \ . \qquad (3.21)$$

We then get that

$$I_2(h,z) = e^{-\frac{i}{h}(\frac{1}{2}b(z)^2 - V(b(z),z))} I_1(h,z) \qquad (3.22)$$

is a C^∞ function of h, and

$$I(h) = \tilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}z^2} e^{\frac{i}{2h}b(z)^2 - \frac{i}{h}V(b(z),z)} I_2(h,z)dz \ , \qquad (3.23)$$

so that $I(h)$ is given by a finite dimensional oscillatory integral.

Now by theorem 2.1 we have that

$$I_2(0,z) = |1 - d_1{}^2V(b(z),z)|^{-\frac{1}{2}} g(b(z),z) \ , \qquad (3.24)$$

and by the corollaries 2.3 and 2.4 and lemma 3.1 we have

$$|I_2(\lambda^2h,z) - I_2(0,z)| \le |h|(1-\lambda^{-2}\int e^{\sqrt{2}\lambda|\beta|} d|\mu|(\beta,\gamma))^{-2} \int e^{\sqrt{2}\lambda|\beta|} d|\nu|(\beta,\gamma),$$

so that

$$|I_2(h,z) - I_2(0,z)| \le \lambda^{-2}|h|(1-\lambda^{-2}\int e^{\sqrt{2}\lambda|\beta|} d|\mu|(\beta,\gamma))^{-2} \int e^{\sqrt{2}\lambda|\beta|} d|\nu|(\beta,\gamma). \qquad (3.25)$$

Hence $I_2(h,z) - I_2(0,z)$ tends to zero uniformly in z.

Now there exists, as we shall see, a partition $\varphi_i(z)$ of the identity in the finite dimensional space $\mathcal{H}_2$ by smooth functions of compact support, such that

$$I(h) = \sum_{i=1}^{\infty} \widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}z^2} e^{\frac{i}{2h}b(z)^2 - \frac{i}{h}V(b(z),z)} \varphi_i(z) I_2(h,z)dz. \quad (3.26)$$

This is in fact a consequence of the following lemmas 3.3 and 3.4.

Lemma 3.3.

Let $\mathcal{H}$ be a finite dimensional real Hilbert space and $f \in \mathcal{F}(\mathcal{H})$, i.e. $f(x) = \int e^{i\alpha\cdot x} d\mu(\alpha)$ for some bounded complex measure μ on $\mathcal{H}$ and let $\varphi \in \mathcal{S}(\mathcal{H})$ be such that $\varphi(0) = 1$, then

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} f(x)dx = \lim_{\epsilon\to 0} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} \varphi(\epsilon x)f(x)dx$$

$$= \lim_{\epsilon\to 0}(2\pi i)^{-n/2} \int_{\mathcal{H}} e^{\frac{i}{2}x^2} \varphi(\epsilon x)f(x)dx ,$$

where $n = \dim \mathcal{H}$.

Proof: Let $\varphi(x) = \int_{\mathcal{H}} e^{i\alpha x} \hat{\varphi}(\alpha)d\alpha$, then

$$\varphi(\epsilon x) = \int_{\mathcal{H}} e^{i\alpha\cdot\epsilon x} \hat{\varphi}(\alpha)d\alpha = \epsilon^{-n} \int e^{i\alpha x} \hat{\varphi}(\tfrac{1}{\epsilon}\alpha)d\alpha .$$

Hence

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} \varphi(\epsilon x)f(x)dx = \epsilon^{-n} \int_{\mathcal{H}} e^{-\frac{i}{2}(\alpha+\beta)^2} \hat{\varphi}(\tfrac{1}{\epsilon}\alpha)d\alpha d\mu(\beta)$$

$$= \int_{\mathcal{H}} e^{-\frac{i}{2}(\epsilon\alpha+\beta)^2} \hat{\varphi}(\alpha)d\mu(\beta)d\alpha ,$$

which by dominated convergence tends to

$$\int_{\mathcal{H}} e^{-\frac{i}{2}\beta^2} d\mu(\beta) \cdot \int \hat{\varphi}(\alpha)d\alpha = \int_{\mathcal{H}} e^{-\frac{i}{2}\beta^2} d\mu(\beta) .$$

This proves the first identity of the lemma. The second identity follows from the facts that $f(x)$ is bounded, so that $\varphi(\epsilon x)f(x)$ is in L_1, and one has the identity of the normalized and Riemann integral on $\mathcal{S}(\mathcal{H})$ as proved in ref.[10] proposition 2.1. By the continuity of these integrals in $\mathcal{F}(\mathcal{H})$ and $L_1(\mathcal{H})$ respectively this identity also holds for functions in $L_1(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. This then proves the lemma. $\square$

Lemma 3.4.

There is a partition $\varphi_i(x)$ of the identity of the finite dimensional real Hilbert space $\mathcal{H}$ by smooth functions of compact support such that for any $f \in \mathcal{F}(\mathcal{H})$

$$\tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} f(x)dx = \sum_{i=1}^{\infty} \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2}x^2} \varphi_i(x)f(x)dx .$$

Proof: Take $\psi(x)$ such that $\psi(x)$ is a smooth function $0 \leq \psi(x) \leq 1$ and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 , \end{cases}$$

and let $\psi_n(x) = \psi(\frac{1}{n}x)$ and define $\varphi_n(x) = \psi_n(x) - \psi_{n-1}(x)$ for $n > 1$ and $\varphi_1(x) = \psi_1(x)$.

Then obviously $\varphi_n(x)$ is a partition of the unit by smooth functions of compact support and since $\sum_{i=1}^{n} \varphi_i(x) = \psi_n(x)$ this lemma follows from the previous one. $\square$

As a consequence of this lemma we have verified (3.26). Hence to study the asymptotic behavior of $I(h)$ we may consider the asymptotic behavior of each term in (3.26). Now for each term the integrand is in $\mathcal{F}(\mathcal{H}_2) \cap L_1(\mathcal{H}_2)$, so the normalized integrals are actually Riemann integrals and we may therefore use the classical theory of oscillatory integrals.

Since the solutions of the equation

$$d_2V(b(z),z) = z \tag{3.27}$$

form a discrete set, we may assume that only one solution of (3.27) is contained in the support of each $\varphi_i(z)$, if needed by writing each φ_i in (3.26) as a finite sum of positive smooth functions. Hence it is enough to consider integrals of the form

$$I(h,\psi) = \widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h} z^2} e^{\frac{i}{2h} b(z)^2 - \frac{i}{h} V(b(z),z)} \psi(z) I_2(h,z) dz \quad (3.28)$$

where ψ is of compact support and only one solution of (3.27) is contained in the support of ψ .

By the corollaries 2.3 and 2.4 and lemma 3.1 we have that

$$|I_2(h,z) - \sum_{m=o}^{N} \frac{h^m}{m!} I_2^{(m)}(0,z)| \leq |h|^{N+1} \lambda^{-2(N+1)} (N+1)!$$

$$\cdot (1 - \lambda^{-2} \int e^{\sqrt{2}\lambda|\beta|} d|\mu|(\beta,\gamma))^{-N-2} \int e^{\sqrt{2}\lambda|\beta|} d|\nu|(\beta,\gamma) .$$

Hence, up to terms of order $|h|^{N+1}$, (3.28) may be written as

$$\sum_{m=o}^{N} \frac{h^m}{m!} \widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h} z^2} e^{\frac{i}{2h} b(z)^2 - \frac{i}{h} V(b(z),z)} \psi(z) I_2^{(m)}(0,z) dz \ . \quad (3.30)$$

It is therefore enough to study the asymptotic behavior of the integrals

$$\widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h} z^2} e^{\frac{i}{2h} b(z)^2 - \frac{i}{h} V(b(z),z)} \psi(z) I^{(m)}(0,z) dz \quad (3.31)$$

and we know from the classical theory that (3.31) is a C^∞ function of h if the critical points of the exponent are non degenerate. I.e., setting $\varphi(z) = \frac{1}{2} z^2 + \frac{1}{2} b(z)^2 - V(b(z),z)$, we have that the question is whether the solutions of $d\varphi(z) = 0$ are discrete and whether $d^2\varphi(z_o)$ is a non degenerate matrix for any solution of $d\varphi(z_o) = 0$.

Now

$$d\varphi(z) = z + b(z)\cdot db(z) - d_1V(b(z),z)db(z) - d_2V(b(z),z) \quad (3.32)$$

and, since $b(z) = d_1V(b(z),z)$, we have that

$$d\varphi(z) = z - d_2V(b(z),z) \ , \qquad (3.33)$$

so that the critical points are the solutions of (3.27). Hence there exists only one critical point on the support of ψ . Let c be this point and set $b = b(c)$, and $a = (b,c)$ then we see that a is a solution of $dV(x) = x$.

Let us now compute $d^2\varphi(c)$. From (3.33) we have

$$d^2\varphi(z) = 1 - d_2d_1V(b(z),z)db(z) - d_2{}^2V(b(z),z) \ . \qquad (3.34)$$

Since $b(z) = d_1V(b(z),z)$ we get that

$$db(z) = d_1{}^2V(b(z),z)db(z) + d_1d_2V(b(z),z) \ . \qquad (3.35)$$

As a consequence of lemma 3.1, $1 - d_1{}^2V(b(z),z)$ has a bounded inverse, so that

$$db(z) = (1 - d_1{}^2V(b(z),z))^{-1}d_1d_2V(b(z),z) \ . \qquad (3.36)$$

Hence

$$d^2\varphi(c) = 1 - d_2d_1V(a)(1 - d_1{}^2V(a))^{-1}d_1d_2V(a) - d_2{}^2V(a) \ . (3.37)$$

Assume now that $\zeta \in \mathcal{H}_2$ and $d^2\varphi(c)\zeta = 0$, then, with

$$\eta = (1 - d_1{}^2V(a))^{-1}d_1d_2V(a)\zeta \qquad (3.38)$$

and $\xi = (\eta, \zeta)$, we have that

$$(1 - d^2V(a))\xi = 0 \ . \qquad (3.39)$$

This is so because (3.39) is equivalent with

$$(1 - d_1{}^2V(a))\eta + d_1d_2V(a)\zeta = 0$$

$$d_2d_1V(a)\eta + (1 - d_2{}^2V(a))\zeta = 0 \ .$$

The first equation gives, since $(1 - d_1{}^2V(a))$ is non degenerate, the equation (3.38) and the other then gives (3.37).

On the other hand if, for some $\xi \in \mathcal{H}$, $(1 - d^2V(a))\xi = 0$ then $d^2\varphi(c)\zeta = 0$, where ζ is the projection of ξ on $\mathcal{H}_2$. So that the condition that $d^2\varphi(c)$ is non degenerate is equivalent with the condition that $1 - d^2V(a)$ is non degenerate. Moreover we easely get from (3.37) that

$$\mathrm{Det}(d^2\varphi(c)) = \mathrm{Det}(1 - d_1{}^2V(a))^{-1} \cdot \mathrm{Det}(1 - d^2V(a)) \ . \qquad (3.40)$$

Hence we have that, if $1 - d^2V(a)$ is non degenerate, then $I(h,\psi)$ is a C^∞ function of h, and we can find its asymptotic expansion at $h = 0$ by the standard method of stationary phase, see for instance Hörmander ref. [3],1).

Up to now we have assumed $V(x)$ and $g(x)$ to be gentle in the sense of (3.2) and (3.6) respectively. However we see from the proofs above that everything except the discreteness of the solutions of $dV(x) = x$ carries through under a weaker condition which we could call gentle up to finite codimension. The precise condition is given in the following theorem.

Theorem 3.1.

Let $\mathcal{H}$ be a real separable Hilbert space, and V and g in $\mathcal{F}(\mathcal{H})$, i.e. there are bounded complex measures on $\mathcal{H}$ such that

$$V(x) = \int_{\mathcal{H}} e^{\,ix\alpha}\, d\mu(\alpha) \quad \text{and} \quad g(x) = \int_{\mathcal{H}} e^{\,ix\alpha}\, d\nu(\alpha) \ .$$

Let us assume V and g C^∞, i.e. all moments of μ and ν

exist. Moreover we assume $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\dim \mathcal{H}_2 < \infty$, and if $d\mu(\beta,\gamma)$, $d\nu(\beta,\gamma)$ are the measures on $\mathcal{H}_1 \times \mathcal{H}_2$ given by μ and ν, then there is a λ such that $\|\mu\| < \lambda^2$ and

$$\int_{\mathcal{H}} e^{\sqrt{2}\lambda|\beta|} \, |d\mu|(\beta,\gamma) < \infty \quad \text{and} \quad \int_{\mathcal{H}} e^{\sqrt{2}\lambda|\beta|} \, d|\nu|(\beta,\gamma) < \infty .$$

Then if the equation $dV(x) = x$ has only a finite number of solutions $x_1, \dots, x_n$ on the support of the function $g(x)$, such that none of the operators $1 - d^2V(x_i)$ has zero as an eigenvalue, then the function

$$I(h) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} \, e^{-\frac{i}{h}V(x)} \, g(x)dx$$

is of the following form

$$I(h) = \sum_{k=1}^{n} e^{\frac{i}{h}(\frac{1}{2}x_k^2 - V(x_k))} \, I_k^*(h) ,$$

where $I_k^*(h)$ is a C^∞ function of h such that

$$I_k^*(0) = e^{\frac{i\pi}{2}n_k} |\mathrm{Det}(1 - d^2V(x_k))|^{-\frac{1}{2}} g(x_k)$$

where n_k is the number of eigenvalues of $d^2V(x_k)$ which are larger than 1.

If $V(x)$ is gentle on $\mathcal{H}$ i.e. satisfies (3.2), then the solutions of the equation $dV(x) = x$ have no limit points.

Proof: By applying lemma 3.1 to the real separable Hilbert space $\mathcal{H}_1$ we get a decomposition of $\mathcal{H} = \mathcal{H}_1' \oplus \mathcal{H}_2'$ with $\mathcal{H}_2 \subseteq \mathcal{H}_2'$ and $\mathcal{H}_2'$ finite dimensional such that, with $x = y' \oplus z'$, $V(x) = V(y',z')$ is, as a function of $y' \in \mathcal{H}_1'$, gentle and small uniformly in $z \in \mathcal{H}_2'$. In fact we have that

$$\frac{1}{\lambda^2}\int_{\mathcal{H}_1'} e^{\sqrt{2}\lambda|\beta'|}\, d|\mu_{z'}|(\beta') \leq \frac{1}{\lambda^2}\int_{\mathcal{H}} e^{\sqrt{2}\lambda|\beta'|}\, d|\mu|(\beta',\gamma') < 1 .$$

So, if necessary by using the decomposition $\mathcal{H} = \mathcal{H}_1' \oplus \mathcal{H}_2'$ instead of $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we may assume that, with the notations of the theorem,

$$\frac{1}{\lambda^2}\int_{\mathcal{H}_1} e^{\sqrt{2}\lambda|\beta|}\, d|\mu_z|(\beta) \leq \frac{1}{\lambda^2}\int_{\mathcal{H}} e^{\sqrt{2}\lambda|\beta|}\, d|\mu|(\beta,\gamma) < 1 \; . \tag{3.41}$$

And from this it follows in the same way as in lemma 3.1 that the equation

$$d_1V(y,z) = y \tag{3.42}$$

has a unique solution $y = b(z)$. Moreover from (3.41) we get that

$$\|d_1{}^2V(y,z)\| \leq \frac{1}{\lambda^2}\int_{\mathcal{H}} e^{\sqrt{2}\lambda|\beta|}\, d|\mu|(\beta,\gamma) < 1 \; , \tag{3.43}$$

hence $z \to b(z)$ is a smooth mapping of $\mathcal{H}_1$ into $\mathcal{H}_2$.
Now by using the Fubini theorem (Prop. 2.4 of ref. [10]) on $I(h)$ we have the formula (3.23). Since there are only a finite number of critical points $x_1,\dots,x_n$ on the support of $g(x)$ there will only be a finite number of critical points $z_1,\dots,z_m$ with $x_k = (b(z_k),z_k)$ and z_k are the solutions of

$$d_2V(b(z),z)) = z \, , \tag{3.44}$$

because any solution of (3.44) gives rise to a solution of $dV(x) = x$ of the form $x = (b(z),z)$. Hence we have only a finite number of critical points for the exponent in (3.23).
We may therefore choose a finite partition $\psi_1,\dots,\psi_n$ of the unit in $\mathcal{H}_2$ such that ψ_k are identically one in a neighbourhood of z_k and zero in a neighbourhood of z_j with $j \neq k$

and the asymptotic expansion for

$$I(h,\psi_k) = \widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}z^2} e^{\frac{i}{2h}b(z)^2 - \frac{i}{h}V(b(z),z)} I_z(h,z)\psi_k(z)dz \quad (3.45)$$

follows in the same way as the asymptotic expansion of $I(h,\psi)$ given in (3.28).

Utilizing the identity (3.40) for the determinants and the fact that $d^2\varphi(c)$ and $1 - d^2V(a)$ with $a = (b(c),c)$ have the same number of negative eigenvalues, since $\|d_1{}^2V(a)\| < 1$, the rest of the theorem then follows from the classical theory of oscillating integrals, see for instance ref.[3],1) section 1.

To see that $d^2\varphi(c)$ and $1 - d^2V(a)$ have the same number of negative eigenvalues we use the determinant identity (3.40) with $tV(x)$ instead of $V(x)$. We then have that

$$d^2\varphi_t(c) = 1 - t^2 d_2 d_1 V(a)(1 - td_1{}^2V(a))^{-1} d_1 d_2 V(a) - td_2{}^2V(a) \quad (3.46)$$

and the determinant equation takes the form

$$\mathrm{Det}(d^2\varphi_t(c)) = \mathrm{Det}(1 - td_1{}^2V(a))^{-1}\,\mathrm{Det}(1 - td^2V(a)). \quad (3.47)$$

Now $\mathrm{Det}\,(1 - td_1{}^2V(a))^{-1} > 0$ for $0 \leq t \leq 1$ so that $\mathrm{Det}(d^2\varphi_t(c))$ and $\mathrm{Det}(1 - td^2V(a))$ have the same zeros with the same multiplicities, moreover the sign of the derivatives with respect to t at the zeros must be the same. So if the negative eigenvalues of $1 - d^2V(a)$ are all distinct the number of negative eigenvalues of $d^2\varphi(c)$ and $1 - d^2V(a)$ must be the same, because then all zeros of the two functions are simple. If $1 - d^2V(a)$ has multiple negative eigenvalues, there is an operator close in trace norm to $1 - d^2V(a)$ which has negative eigenvalues which are all distinct, and the result follows from the continuous dependence of $d^2\varphi(c)$ on $1 - d^2V(a)$ and the continuity of the determinant with respect to the trace norm. This then proves the theorem.

4. Oscillatory Integrals and Lagrange Immersions.

In the previous sections we have studied the behavior of the integrals of the form

$$I(h) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x)} g(x)dx \tag{4.1}$$

where $\mathcal{H}$ is a real separable Hilbert space and V and g are in $\mathcal{F}(\mathcal{H})$, and we proved under additional regularity assumptions on V and g (theorem 3.1) that, if the phase function $\frac{1}{2}x^2-V(x)$ has only non degenerate critical points, then $I(h)$ is a C^∞ function of h and its asymptotic expansion at $h = 0$ depends only on the derivatives of V and g at these critical points. We know from the case of finite dimensional $\mathcal{H}$ that, if some critical point of $\frac{1}{2}x^2 - V(x)$ is degenerate, then $I(h)$ will not tend to a limit as $h \to 0$. In the case of finite dimensional $\mathcal{H}$ however, one studies the situation of degenerate critical points for the phase function by letting V and g depend on some additional parameter $y \in R^k$. This is very natural because we know by the Morse theorem that the set of functions $V(x)$ such that $\frac{1}{2}x^2 - V(x)$ has only non degenerate critical points form an open and dense set in the space of C^∞-functions. In fact the complement is in a natural sense of codimension 1. Hence the case of degenerate critical points is unstable in the sense that degenerate critical points will disappear under arbitrary small perturbations, i.e. small in the sense of the C^∞ topology.

On the other hand if V depends on additional parameters $y \in Y = R^k$, we know by Thom's transversality theorem (ref.[9]) that there is an open dense set of functions $y \to V(\cdot,y)$ from

$Y = R^k$ into $C^\infty(\mathcal{H})$ such that for each function $y \to V(\cdot,y)$ in this set, the function induced by $y \to \frac{1}{2}x^2 - V(x,y)$ in the jet bundle over $\mathcal{H}$ intersects the singular manifolds in this jet bundle transversally, hence they intersect only the singular manifolds of codimension at most k. Moreover these intersections are transversal hence stable, i.e. all nearby functions also intersect the same singular manifolds. For this reason it is only natural to study singularities of codimension k by studying k-dimensional families. We shall see in what follows that this is the situation also when $\mathcal{H}$ is a infinite dimensional real separable Hilbert space.

So in what follows we shall consider oscillatory integrals of the form

$$I(h,y) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2} e^{-\frac{i}{h}V(x,y)} g(x,y)dx \tag{4.2}$$

where we assume that $y \in Y = R^k$ and that $y \to V(\cdot,y)$ and $y \to g(\cdot,y)$ are C^∞-functions from Y into $\mathcal{F}(\mathcal{H})$ in the strong topology, so that $I(h,y)$ is a C^∞-function in y for $h \neq 0$.

As in the case of finite dimensional $\mathcal{H}$ (see

we study (4.2) by integrating $I(h,y)$ with an oscillatory function $e^{-\frac{i}{h}\psi(y)}\chi(y)$ where ψ and χ are C^∞-functions and χ has compact support. Hence let

$$I(h,\psi) = (2\pi ih)^{-\frac{h}{2}} \int_Y e^{-\frac{i}{h}\psi(y)} \chi(y)I(h,y)dy . \tag{4.3}$$

We shall assume that $V(x,y)$ and $g(x,y)$ satisfy as functions of x the conditions of theorem 3.1 uniformly in g on the support of χ. By the Fubini theorem for the Fresnel integral

(Prop.2.4 ref.[10]) we may consider (4.3) as the normalized integral over $\mathcal{H} \oplus Y$, where Y is equipped with the scalar product $\frac{1}{h} y \cdot y$. We shall therefore apply theorem 3.1 to (4.3) . We see that the conditions of theorem 3.1 are satisfied if the phase function

$$\varphi(x,y) = \tfrac{1}{2}x^2 - V(x,y) - \psi(y) \tag{4.4}$$

has only a finite number of regular critical points on the support of $\chi(y)g(x,y)$.

The critical points of (4.2) are the solutions of the equation $x = d_1V(x,y)$. Hence we define the singular locus S_V by

$$S_V = \{(x,y) \in \mathcal{H} \oplus Y \; ; \quad x = d_1V(x,y)\} \, . \tag{4.5}$$

That (4.4) has only regular critical points is the condition that on the set of solutions of

$$x = d_1V(x,y) \quad \text{and} \quad -d_2V(x,y) = d\psi(y) \tag{4.6}$$

the operator

$$d^2\varphi = \begin{pmatrix} 1 - d_{11}V & -d_{12}V \\ -d_{21}V & -d_{22}(\psi + V) \end{pmatrix} , \tag{4.7}$$

where $d_{12}V = d_2d_1V$ etc., is non singular. By the assumptions in theorem 3.1 V is gentle up to finite codimension, which implies that $d_{11}V$ is compact. Hence by the Fredholm alternative $d^2\varphi$ is non singular if and only if it is surjective. But then the first component must be surjective so that

$$dd_1\varphi = (1 - d_{11}V, -d_{12}V) \tag{4.8}$$

is surjective as a operator from $\mathcal{H} \oplus R^k$ into $\mathcal{H}$. Since $d_{11}V$

is compact, we get by the Fredholm alternative that the kernel of the operator (4.8) is k dimensional. We have namely that this kernel is given by

$$(1 - d_{11}V)x - d_{12}Vy = 0 \ . \tag{4.9}$$

If 1 is not an eigenvalue of $d_{11}V$, then we solve (4.9) by taking $x = (1 - d_{11}V)^{-1} d_{12}Vy$, so that the kernel of (4.8) is k-dimensional. If however, 1 is an eigenvalue of $d_{11}V$ with the finite dimensional eigenspace $\mathcal{H}_1$, we get from the fact that (4.8) is surjective that

$$\mathcal{H}_1 \subseteq (d_{12}V)Y \ . \tag{4.10}$$

From (4.10) it follows that if $l = \dim \mathcal{H}_1$, then $l \leq k$. Let P_1 be the orthogonal projection onto $\mathcal{H}_1$, then (4.9) implies that

$$P_1(d_{12}V)y = 0 \tag{4.11}$$

for all solutions (x,y) of (4.9). Now if $l = k$, we have from (4.10) that $\mathcal{H}_1 = (d_{12}V)Y$, and, since $l = k$, the range $\mathcal{H}_1$ of $d_{12}V$ has the same dimension as the domain Y so that the dimension of the kernel of $d_{12}V$ must be zero. So for $l = k$ (4.11) implies that $y = 0$, hence $d_{12}Vy = 0$ and the solutions of (4.9) are of the form $(x,0)$ with $x \in \mathcal{H}_1$, which again is k dimensional.

Now if $l < k$ let $U \subseteq y$ be the $k-l$ dimensional subspace of solutions of (4.11). For arbitrary $y \in U$ (4.9) has a solution because of (4.11) and the dimension of the set of solutions is equal to the dimension of $\mathcal{H}_1$ which is l. Hence the dimension of the space of solutions is again k. So we have proved that the dimension of the kernel of (4.8) is always k. Set now $T = dd_1\varphi$ and, for $(x,y) \in S_V$, set

$$A_{(x,y)} = T^*T . \qquad (4.12)$$

Then $A_{(x,y)}$ is a symmetric operator on the Hilbert space $\mathcal{H} \oplus Y$ such that $A-1$ is compact. Moreover $A_{(x,y)}$ depends norm continuously on $(x,y) \in S_V$ and $A_{(x,y)}$ has as eigenspace for the eigenvalue 0 the kernel of $T = dd_1\varphi$. By the theory of regular perturbation (see ref. [27]) of self adjoint operators with discrete spectrum we have that, since the dimension of the eigenspace for the eigenvalue 0 is constant, none of the eigenvalues different from zero approach zero so that zero is an isolated eigenvalue as (x,y) runs over S_V. From this it follows by the standard technique of the perturbation of operators with discrete spectrum, that if $P_{(x,y)}$ is the projection onto the eigenspace of the eigenvalue zero, then $P_{(x,y)}$ is norm continuous in $(x,y) \in S_V$.

However, $T_{(x,y)}$ is the derivative of the function $d_1\varphi = x - d_1V(x,y)$, so that the kernel of $T_{(x,y)}$ is the tangent space of S_V at the point $(x,y) \in S_V$. Hence $P_{(x,y)}$ is the projection in $\mathcal{H} \oplus Y$ onto the tangent space of S_V at $(x,y) \in S_V$. By the argument above it follows that

$$P_{(x,y)} = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{dz}{z - A_{(x,y)}} , \qquad (4.13)$$

where ρ can be taken independent of (x,y) for $(x,y) \in K \subset S_V$, where K is any compact subset of S_V. Since $A_{(x,y)} = T^*T$ is a C^∞-function of (x,y) we get from (4.13) that $P_{(x,y)}$ is a C^∞-function of (x,y). This then gives us that S_V is a smooth manifold in $\mathcal{H} \oplus Y$, and in fact that S_V is a smooth manifold is equivalent with (4.8) being surjective. Phase functions $\frac{1}{2}x^2 - V(x,y)$ which satisfy this condition are called non degenerate. So only for non degenerate phase functions can we hope to find a $\psi(y)$ such that $d^2\varphi$ have only regular singularities. We also observe from the proof above that for a non degenerate phase function we have that the multiplicity of the eigenvalue zero of $1 - d_{11}V$ at $(x,y) \in S_V$ is always smaller or equal to $k = \dim Y$.

Consider now the mapping from $\mathcal{H} \oplus Y \to T^*Y$, where T^*Y is the cotangent bundle of Y , given by

$$(x,y) \to (y, -d_2V(x,y)) \ . \tag{4.14}$$

This mapping has a differential the restriction of which to the space of solutions of (4.9) is injective. This is seen in the following way. $(\Delta x, \Delta y)$ is in the kernel of the differential of (4.14) if

$$(\Delta y, -d_{21}V\ \Delta x - d_{22}V\ \Delta y) = 0$$

so that $\Delta y = 0$ and $d_{21}V\ \Delta x = 0$. Hence if $(\Delta x, \Delta y)$ is also a solution of (4.9), then we have $\Delta x \in \mathcal{H}_1$ and $d_{21}V\ \Delta x = 0$. So we have only to prove that the restriction of $d_{21}V$ to $\mathcal{H}_1$ is injective. But from the fact that (4.8) is surjective it follows that $\mathcal{H}_1 \subseteq (d_{12}V)Y$ and this implies, since $d_{12}V$ is the adjoint of $d_{21}V$, that the restriction of $d_{21}V$ to $\mathcal{H}_1$ is indeed injective. Therefore the restriction of (4.14) to S_V is locally a smooth embedding of S_V in T^*Y since the solutions of (4.9) are the tangent space of S_V . So the restriction of (4.14) to S_V defines an immersion Λ_V of the singular locus S_V

in T^*Y. So that Λ_V is a smooth k-dimensional manifold in T^*Y. It follows as in the finite dimensional case (see [5] section 1.2) that Λ_V is a Lagrange manifold, i.e. the canonical symplectic form on T^*Y vanishes on the tangent spaces of Λ_V. Therefore we say that Λ_V is the Lagrange immersion in T^*Y of the singular locus S_V.

Because the differential of (4.14) is given by

$$(\Delta x,\Delta y) \to (\Delta y,\ d_{21}V\ \Delta x + d_{22}V\ \Delta y)\ , \tag{4.15}$$

and the tangent space of S_V is given by

$$(1-d_{11}V)\Delta x - d_{12}V\ \Delta y = 0\ , \tag{4.16}$$

we have to prove that the image of the subspace given by (4.16) by the mapping (4.15) is symplectic.

Since $Y = R^k$ there is a natural identification of T^*Y with $Y \oplus Y$ and of the tangent space of T^*Y with $Y \oplus Y$. Under this identification the canonical symplectic form is given by

$$\langle \xi,\eta\rangle = \alpha\cdot\delta - \beta\cdot\gamma\ , \tag{4.17}$$

where $\xi = (\alpha,\beta)$ and $\eta = (\gamma,\delta)$. Let now ξ be the image of $(\Delta x_1,\Delta y_1)$ and η the image of $(\Delta x_2,\Delta y_2)$ with $(\Delta x_i,\Delta y_i)$ satisfying (4.16), then

$$\begin{aligned}\langle \xi,\eta\rangle &= \Delta y_1\cdot(d_{21}V\Delta x_2 + d_{22}V\Delta y_2)\\ &\quad - \Delta y_2\cdot(d_{21}V\Delta x_1 + d_{22}V\Delta y_1)\\ &= d_{12}V\Delta y_1\cdot\Delta x_2 - d_{12}V\Delta y_2\cdot\Delta x_1\ ,\end{aligned}$$

which by (4.16) is equal to

$$\Delta x_1(1-d_{11}V)\Delta x_2 - \Delta x_1\cdot(1-d_{11}V)\Delta x_2\ .$$

Hence we have that $\langle \xi, \eta \rangle = 0$.

Now we see that in order that $d^2\varphi$ given by (4.7) should be non singular, we must have that the restriction of

$$dd_2\varphi = (-d_{21}V, -d_{22}(\psi+V)) \tag{4.18}$$

to the kernel of (4.8) should be injective. So we get the condition on ψ that if $(\Delta x, \Delta y)$ satisfies

$$\begin{aligned} &d_{22}\psi\Delta y = -d_{21}V\Delta x - d_{22}V\Delta y \\ \text{and} \qquad & \\ &(1-d_{11}V)\Delta x - d_{12}V\Delta y = 0 \end{aligned} \tag{4.19}$$

then $\Delta x = \Delta y = 0$.

Now Λ_V is the image by (4.14) of the solutions of $x = d_1V(x,y)$, so the other equation (4.6), that is $d\psi(y) = -d_2V(x,y)$, says just that the graph $(y, d\psi(y))$ intersects Λ_V at the points in Λ_V which are images of points $(x,y) \in S_V$ that satisfy (4.6). Now $(y, d\psi(y))$ is obviously a Lagrange manifold of dimension k and the condition given by (4.19) is just that the Lagrange manifold $(y, d\psi(y))$ intersects Λ_V transversally. Because both are k-dimensional and T^*Y is of dimension $2k$, it is enough to prove that the intersection of their tangent spaces contains only the zero vector. The tangent space of Λ_V is given by (4.15) and (4.16) on the form

$$(\Delta y,\ d_{21}V\Delta x + d_{22}V\Delta y)\ , \tag{4.20}$$

where $(\Delta x, \Delta y)$ satisfies (4.16) .

On the other hand the tangent space of $(y, d\psi(y))$ is given by all vectors of the form $(\Delta y,\ d_{22}\psi\Delta y)$. Hence the intersection of the two tangent spaces is exactly given by the two equations

(4.19). Therefore the original condition that (4.19) has only the trivial solution is equivalent to the condition of transversal intersection.

Hence the final condition on ψ is just that $(y, d\psi(y))$ and Λ_V are transversal, which is a condition given entirely by the Lagrange immersion Λ_V in the tangent bundle T^*Y of the finite dimensional parameterspace Y.

Let us now assume that the two Lagrange manifolds $(y, d\psi(y))$ and Λ_V intersect transversally and that only one point of the intersection is contained in the support of χ. We denote this point by (y_o, ξ_o). Then by theorem 3.1 we have that, for $I(h,\psi)$ given by (4.3),

$$I(h,\psi) = |\mathrm{Det}\ d^2\varphi|^{-\frac{1}{2}}\ e^{i\frac{\pi}{2}n(d^2\varphi)}\ e^{\frac{i}{h}(\frac{1}{2}x_o{}^2 - V(x_o,y_o))} g(x_o,y_o)\chi(g_o) + O(h), \tag{4.21}$$

where we have assumed that $\psi(y_o) = 0$ and that only one point $(x_o, y_o) \in S_V$ is in the preimage of $(y_o, \xi_o) \in \Lambda_V$ by the Lagrange immersion (4.14). More generally if the Lagrange immersion $S_V \to \Lambda_V \subset T^*Y$ is proper, then instead of (4.21) we get the sum over all preimage (x_o, y_o) of (y_o, ξ_o). Remark that $n(d^2\varphi)$ is the number of negative eigenvalues of $d^2\varphi(x_o, y_o)$. Since we have proved that the differential of $S_V \to \Lambda_V$ is injective, we know that locally the mapping is one-to-one, so for local considerations we may take the oscillating factor $S = \frac{1}{2}x_o{}^2 - V(x_o, y_o)$ in (4.21) to be a function on Λ_V. Since $x_o = d_1V(x_o, y_o)$ and $-d_2V(x_o, y_o) = d\psi(y_o) = \xi_o$ for any $(x_o, y_o) \in S_V$ we see that

$$dS = \xi \cdot dy , \tag{4.22}$$

where $\xi \cdot dy$ is the restriction to Λ_V of the canonical one-form

$\xi \cdot dy = \sum_{i=1}^{k} \xi_i dy_i$ in the cotangent bundle T^*Y. Hence up to an additive constant the oscillating factor $S = \frac{1}{2}x_o{}^2 - V(x_o, y_o)$ depends only on the Lagrange immersion $\Lambda_V \subset T^*Y$.

Now $|\mathrm{Det}\, d^2\varphi|$ and $n(d^2\varphi)$ depend of course on the function $\psi(y)$ but only through $d^2\psi(y_o)$, i.e. on the direction of the Lagrange manifold $(y, d\psi(y))$ at the point of intersection (y_o, ξ_o). We now write (4.7) as

$$d^2\varphi = \binom{A_1}{A_2}$$

where A_1 is a mapping from $\mathcal{H} \oplus Y$ into $\mathcal{H}$ and A_2 is into Y. Since $d^2\varphi$ is non degenerate, we have that both the following two mappings are non degenerate

$$\begin{aligned} A_1 &: (\mathcal{H} \oplus Y) \ominus \ker A_1 \to \mathcal{H} \\ A_2 &: \ker A_1 \to Y . \end{aligned} \tag{4.23}$$

Now let U be an orthogonal transformation that maps Y onto $\ker A_1$ and hence $\mathcal{H}$ onto the orthogonal complement of $\ker A_1$. That there is such an orthogonal transformation follows from the fact that $\ker A_1 = T_{(x_o, y_o)} S_V$, which we have shown to have the same dimension as Y. Then $d^2\varphi \cdot U$ maps $\mathcal{H}$ into $\mathcal{H}$ and Y into Y. Hence $d^2\varphi \cdot U = B_1 \oplus B_2$, where $B_i = A_i U$ restricted to $\mathcal{H}$ and Y respectively, so that $\mathrm{Det}\, d^2\varphi \cdot U = \mathrm{Det}\, B_1 \cdot \mathrm{Det}\, B_2$. Since Det U is equal to ± 1 we therefore have

$$|\mathrm{Det}\, d^2\varphi| = |\mathrm{Det}\, B_1| \cdot |\mathrm{Det}\, B_2| . \tag{4.24}$$

Because of $B_i = A_i U$ we have, U being orthogonal, that $|\mathrm{Det}\, B_i|$ is equal to the volume deformation of the transformations (4.23). In particular $\mathrm{Vol}_{S_V} = |\mathrm{Det}\, B_1|^{-1}$ is the volume in the tangent space of S_V , which in the finite dimensional case is the volume given by $\delta(x - d_1V(x,y))$ i.e. the restriction of the canonical volume in $\mathcal{H} \oplus Y$ to the submanifold S_V . Vol S_V is independent of ψ and by the local isomorphism $S_V \to \Lambda_V$ it induces a volume Vol_V on Λ_V .

On the other hand $|\mathrm{Det}\ B_2|$ is the volume deformation by the second transformation (4.23), the differential of which is given by

$$(\Delta x, \Delta y) \to (0, - d_{21}V\Delta x - d_{22}(\psi + V)\Delta y) \ . \qquad (4.25)$$

Now the differential of the immersion $S_V \to \Lambda_V$ is given by the restriction to the tangent space of S_V of the mapping

$$(\Delta x, \Delta y) \to (\Delta y, - d_{21}V\,\Delta x - d_{22}V\Delta y) \ . \qquad (4.26)$$

Consider now the mapping $T_{(x,y)}S_V \to T_y^*Y$ given by composing the differential of the Lagrange immersion $S_V \to \Lambda_V$ with the projection along the direction given by the tangent space of $(y, d\psi(y))$ of the tangent of Λ_V onto the fiber T_y^*Y . This projection is obtained by adding to the right hand side of (4.26) an element of the form $(\Delta y_1, d^2\psi \cdot \Delta y_1)$ such that the result is in the fiber T_y^*Y . Hence $\Delta y_1 = -\Delta y$ and the differential of $T_{(x,y)}S_V \to T_y^*Y$ is given by

$$(\Delta x, \Delta y) \to (0, - d_{21}V\,\Delta x - d_{22}(V + \psi)\,\Delta y) \ ,$$

which is identical with (4.25). So we see that $\mathrm{Vol}_\psi = |\mathrm{Det}\ B_2|$ is simply the volume deformation of the mapping from the tangent space at (x,y) of S_V into the fiber T_y^*Y , by first taking the differential of the Lagrange immersion $S_V \to \Lambda_V$ and then projecting the tangent of Λ_V onto the fiber T_y^*Y along the direction given by the tangent of $(y, d\psi(y))$. We have thus that

$$|\mathrm{Det}\; d^2\varphi| = (\mathrm{Vol}_V)^{-1}\cdot(\mathrm{Vol}_\psi) \;. \tag{4.27}$$

Moreover for the number of negative eigenvalues of $d^2\varphi$ we have the following lemma

Lemma 4.1.

$$n(d^2\varphi) = n(1-d_{11}V) + n(A_\psi) \;,$$

where $n(A)$ is the number of negative eigenvalues of $1-A$ for a compact operator A with the convention that each zero eigenvalue is counted as $\frac{1}{2}$, and A_ψ is the symmetric transformation of T_y^*Y given by $A_\psi\xi = \Delta y$, where Δy is given by the unique solution of

$$\begin{pmatrix} 1-d_{11}V & -d_{12}V \\ -d_{21}V & -d_{22}(V+\psi) \end{pmatrix}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = \begin{pmatrix} 0 \\ \xi\end{pmatrix} \;.$$

Proof: The proof of this lemma goes exactly through in the same way as in the case of finite dimensional $\mathcal{H}$, and in this case it is the lemma 1.2.2 of ref. [5]. □

In particular we get that with two different ψ_1 and ψ_2 giving the same point of intersection (y_o,ξ_o) we have

$$n(d^2\varphi_1) - n(d^2\varphi_2) = n(A_{\psi_1}) - n(A_{\psi_2}) = \sigma(M_1,M_2;L_{\psi_1},L_{\psi_2}) \;, \tag{4.28}$$

where $M_1 = T^*_{y_o}Y$, i.e. the fiber over y_o in the cotangent bundle T^*Y, and M_2 is the tangent space of Λ_V at (y_o,ξ_o). L_{ψ_1} and L_{ψ_2} are the tangent spaces of $(y,d\psi_1(y))$ and $(y,d\psi_2(y))$, where σ is defined in ref.[3],1) section 3.3.

Definition. If $\lambda_1,\lambda_2,\mu_1,\mu_2$ are four Lagrange planes in a symplectic vector space E such that each of the two first is transveral to each of the two last, we have

$$\sigma(\lambda_1,\lambda_2;\mu_1,\mu_2) = \langle\gamma,\alpha_E\rangle,,$$

where γ is a closed curve in $\Lambda(E)$ (the set of Lagrange planes in E) which consists of an arc from μ_1 to μ_2 of Lagrange planes transversal to λ_1 followed by an arc from μ_2 to μ_1 of Lagrange planes transversal to λ_2. Here α_E is the canonical cohomology class studied by Keller, Maslov and Arnold and called the Maslov index by Arnold. For an introduction to the Maslov index we refer the reader to a paper by Arnold [28] which appeared also as Appendix in Maslov's book [2],2).

We have now in fact proved the following theorem.

Theorem 4.1.

If $y \to g(x,y)$ and $y \to \chi(y)$ are $\frac{1}{2}$-densities in Y, then the leading term of the asymptotic expansion of

$$|\mathrm{Vol}_\psi|^{\frac{1}{2}}\, I(h,\psi) = |\mathrm{Vol}_\psi|^{\frac{1}{2}}\,(2\pi i h)^{-\frac{k}{2}} \int_Y e^{-\frac{i}{h}\psi}\,\chi(y) I(h,y)\,dy\ ,$$

where

$$I(h,y) = \tilde{\int_{\mathcal{H}}} e^{\frac{i}{2h}x^2}\, e^{-\frac{i}{h}V(x,y)}\, g(x,y)\,dx\ ,$$

is an element of $\Omega_{\frac{1}{2}}(y_0,\xi_0)\otimes L(y_0,\xi_0)$, where $\Omega_{\frac{1}{2}}(y_0,\xi_0)$ denotes the space of $\frac{1}{2}$-densities on $T_{(y_0,\xi_0)}\Lambda_V$ and $L(y_0,\xi_0)$ is the fiber at (y_0,ξ_0) of the Maslov canonical line bundle L on Λ_V, where we have assumed that V and g satisfy as functions of x the conditions of theorem 3.1 uniformly in y, $(y,d\psi(y))$ intersects Λ_V transversally, the Lagrange immersion $S_V \to \Lambda_V$ is proper, and the phase function $\frac{1}{2}x^2 - V(x,y)$ is non degenerate. Moreover the leading term is given by the sum of the values of the/ /following expression evaluated at the preimages $(x_0,y_0)\in S_V$ of the points (y_o, ξ_o) of intersection between the Lagrange immersion Λ_V and the Lagrange manifold $(y, d\psi(y))$. The expression is

$$|\mathrm{Vol}_V|^{\frac{1}{2}} \, e^{\frac{i\pi}{2} n(1-d_{11}V)} \, e^{\frac{i\pi}{2} n(A_\psi)} \, e^{\frac{i}{h} S} \, g\chi e^{-\frac{i}{h}\psi},$$

where $S(x_o, y_o) = \frac{1}{2} x_o^2 - V(x_o, y_o)$ is given uniquely up to an additive constant by the Lagrange immersion Λ_V. In fact we have, if we locally consider S as a function on Λ_V,

$$dS = \xi \cdot dy .$$

Moreover if we have two different ψ_1 and ψ_2 with the same point of intersection (y_o, ξ_o) then

$$n(A_{\psi_1}) - n(A_{\psi_2}) = \sigma(M_1, M_2; L_{\psi_1}, L_{\psi_2}) ,$$

where M_1 is the cotangent fiber and M_2 the tangent to Λ_V at (y_o, ξ_o) and L_{ψ_i} are the tangents to $(y, d\psi_i(y))$.

Proof: That the leading term transforms as an element of $\Omega_{\frac{1}{2}} \otimes L$ follows in the same way as in the finite dimensional case, and for this case we refer to Duistermaat [5 , section 1] and Hörmander [3], 1) section 3 . The rest up to the formula for $n(A_{\psi_1}) - n(A_{\psi_2})$ is actually proved in what precedes the theorem, while the transformation properties of $n(A_\psi)$ follow in the same way as in the finitedimensional case treated by Hörmander [3], 1), section 3. □

Let now again $V(x,y)$ and $g(x,y)$ satisfy the conditions of theorem 3.1 as functions of $x \in \mathcal{H}$, uniformly for $y \in Y$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $x = x_1 \oplus x_2$ where $\mathcal{H}_1$ is finite dimensional and the functions $V(x_1, x_2, y)$ and $g(x_1, x_2, y)$ satisfy the conditions of theorem 2.1 as functions of x_2 uniformly in (x_1, y).

Let us now define

$$I_1^V(x_1,y,h) = \tilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2h}x_2^2} e^{-\frac{i}{h}V(x_1,x_2,y)} g(x_1,x_2,y)dx_2 \tag{4.29}$$

and

$$I_2^V(x_1,y,h) = e^{-\frac{i}{h}(\frac{1}{2}b(x_1,y)^2 - V(x_1,b(x_1,y),y))} I_1(x_1,y,h), \tag{4.30}$$

where $b(x_1,y) \in \mathcal{H}_2$ is the unique solution of

$$x_2 = d_{x_2}V(x_1,x_2,y) . \tag{4.31}$$

If we assume that $y \to V(x,y)$ and $y \to g(x,y)$ is strongly C^∞ from Y to $\mathcal{F}(\mathcal{H})$ and that the conditions in theorem 3.1 are satisfied uniformly in y we get obviously, in the same way as in the proof of theorem 3.1, that $I_2^V(x_1,y,h)$ is C^∞ in all variables and has an asymptotic expansion in h which is uniform in (x_1,y), and that

$$I_2^V(x_1,y,0) = |1 - d_{x_2}^{\,2} V(x_1,b(x_1,y),y)|^{-\frac{1}{2}} g(x_1,b(x_1,y),y) .$$

Moreover we get that $b : \mathcal{H}_1 \oplus Y \to \mathcal{H}_2$ is C^∞. Hence we have that $I(h,y)$ given by (4.2) also may be written

$$I(h,y) = (2\pi i h)^{-\frac{n}{2}} \int_{\mathcal{H}_1} e^{\frac{i}{2h}x_1^2} \cdot e^{\frac{i}{h}(\frac{1}{2}b^2(x_1,y) - V(x_1,b(x_1,y),y))} I_1(x_1,y,h)dx_1, \tag{4.32}$$

where n is the dimension of $\mathcal{H}_1$. Hence we see that $I(h,y)$ is an oscillatory integral in the classical sense of Hörmander with phase function $\varphi(x_1,y) = \frac{1}{2}x_1^{\,2} + \frac{1}{2}b^2(x_1,y) - V(x_1,b(x_1,y),y)$. It is easy to see that the Lagrange immersion defined by the phase function $\varphi(x_1,y)$ is precisely Λ_V. Utilizing the fact that in the finite dimensional case two germs of non degenerate phase functions $\varphi_1(\alpha_1,y)$ and $\varphi_2(\alpha_2,y)$ define the same class of oscillatory integralsmodulo oscillatory factors of the form $e^{\frac{i}{h}\cdot \text{const.}}$ if and only if they define the same germ of a Lagrange immersion in T^*Y, we get the following theorem

Theorem 4.2.

Let $V_i(x_i,y)$ and $g_i(x_i,y)$ be functions on $\mathcal{H}_1 \oplus Y$ where $\mathcal{H}_i$ are separable real Hilbert spaces, such that $y \to V_i(x_i,y)$ and $y \to g_i(x_i,y)$ are strong C^∞ functions from Y to $\mathcal{F}(\mathcal{H}_1)$ and, as functions of x_i, they satisfy the conditions of theorem 3.1 uniformly in y. Moreover let us assume that the phase functions $x_i^2 - V_i(x_i,y)$ are non degenerate at $(x_i,y_o) \in S_{V_i}$. Then

$$I_i(h,y) = \widetilde{\int_{\mathcal{H}_i}} e^{\frac{i}{2h}x_i^2} e^{-\frac{i}{h}V_i(x_i,y)} g_i(x_i,y)dx_i$$

define the same class of oscillatory integrals modulo oscillatory factors $e^{\frac{i}{h}\text{const}}$ if and only if the germs of the two phase functions $x_i^2 - V_i(x_i,y)$ at (x_i^o,y_o) respectively define the same germs of Lagrange immersions Λ_{V_1} and Λ_{V_2} in T^*Y. That the oscillatory integrals belong to the same class just means that one gets the one from the other by making a smooth y-dependent transformation in the integral.

5. The approach to the classical limit in Quantum Mechanics.

In this section we shall as an illustration apply the theory of the asymptotic behavior of oscillatory integrals on infinite dimensional spaces, as developed in the preceding sections, to the particular case of the Feynman path integral as developed in ref. [10].

So let

$$S = \left(\int_0^t \frac{m}{2}\dot{q}^2(\tau) - V(q(\tau))\right)d\tau \tag{5.1}$$

be the classical action of a classical mechanical system of d degrees of freedom, i.e. $q(\tau)$ is a continuous function from $[0,t]$ to R^d, and $V(q)$ is the potential of the system. We shall assume that the potential V is real and in $\mathcal{F}(R^d)$ i.e.

$$V(q) = \int e^{i\beta q}\, d\nu(\beta) \tag{5.2}$$

where ν is a bounded complex measure and $\overline{\nu(\beta)} = \nu(-\beta)$. We shall also for simplicity of notation take $m = 1$.

Let now $\mathcal{H}$ be the real Hilbert space of continuous functions $\gamma(\tau)$ from $[0,t]$ to R^d such that $\gamma(t) = 0$ and

$$|\gamma|^2 = \int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau \tag{5.3}$$

is finite. In theorem 3.1 of ref [10] we proved that the solution of the Schrödinger equation

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi \tag{5.4}$$

with initial condition $\psi(x,0) = \varphi(x)$ is given by the normalized integral

$$\psi(x,t) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau}\, e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau)+x)d\tau}\, \varphi(\gamma(0)+x)d\gamma \tag{5.5}$$

for $\varphi \in \mathcal{F}(R^d)$. We also write (5.5) shortly as

$$\psi(x,t) = \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{2\hbar}\int_0^t \left|\frac{d\gamma}{d\tau}\right|^2 d\tau} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(\tau))d\tau} \varphi(\gamma(0))d\gamma \qquad (5.6)$$

or

$$\psi(x,t) = \widetilde{\int_{\gamma(t)=x}} e^{\frac{i}{\hbar}S} \varphi(\gamma(0))d\gamma \; . \qquad (5.7)$$

We now introduce the transition amplitude $G_t(x,y)$ such that

$$\psi(x,t) = \int G_t(x,y)\varphi(y)dy \; . \qquad (5.8)$$

Then $G_t(x,y)$ is the kernel of the one parametric unitary group $e^{-\frac{i}{\hbar}tH}$, where $H = -\frac{\hbar^2}{2}\Delta + V$.

By the Fubini theorem (Prop. 2.4 of ref.[10]), we may express $G_t(x,y)$ as a Feynman path integral as follows. Let $\mathcal{H}_0$ be the closed subspace of $\mathcal{H}$ consisting of paths $\gamma(\tau)$ such that $\gamma(0) = 0$. The orthogonal complement of $\mathcal{H}_0$ is then one dimensional and spanned by the path $\eta(\tau) = 1-\frac{\tau}{t}$. We then have that

$$\mathcal{H} = \mathcal{H}_0 \oplus [\eta] \qquad (5.9)$$

so that the decomposition

$$\gamma(\tau) = \tilde{\gamma}(\tau) + \eta(\tau)\cdot\gamma(0) \qquad (5.10)$$

with $\tilde{\gamma} \in \mathcal{H}_0$ is a direct decomposition in orthogonal subspaces. By the Fubini theorem for normalized integrals we then have

$$\psi(x,t) = \widetilde{\int_{R^d}} e^{\frac{i}{2\hbar}|\eta|^2\cdot y^2} \left[\widetilde{\int_{\mathcal{H}_0}} e^{\frac{i}{2\hbar}\int_0^t \left|\frac{d\tilde{\gamma}}{d\tau}\right|^2 d\tau} e^{-\frac{i}{\hbar}\int_0^t V(\tilde{\gamma}+y\eta+x)d\tau} \varphi(x+y)d\tilde{\gamma}\right]dy. \qquad (5.11)$$

Since the integral over R^d is normalized with respect to the quadratic form $\frac{1}{\hbar}|\eta|^2\cdot y^2$, we get by the definition of the normalized integral, remarking that $|\eta|^2 = t^{-1}$,

$$\psi(x,t) = (2\pi i\hbar t)^{-\frac{d}{2}} \int_{R^d} e^{\frac{i}{2\hbar}|\eta|^2 y^2}$$

$$\widetilde{\int_{\mathcal{H}_o}} e^{\frac{i}{2\hbar}\int_o^t |\frac{d\tilde{\gamma}}{d\tau}|^2 d\tau} e^{-\frac{i}{\hbar}\int_o^t V(\tilde{\gamma}+y\eta+x)d\tau} \varphi(x+y)d\tilde{\gamma}\, dy. \quad (5.12)$$

So that

$$G_t^\hbar(x,y) = (2\pi i\hbar t)^{-\frac{d}{2}} e^{\frac{i}{2\hbar}|\eta|^2(y-x)^2}$$

$$\widetilde{\int_{\mathcal{H}_o}} e^{\frac{i}{2\hbar}\int_o^t |\frac{d\tilde{\gamma}}{d\tau}|^2 d\tau} e^{-\frac{i}{\hbar}\int_o^t V(\tilde{\gamma}+(y-x)\eta+x)d\tau} d\tilde{\gamma}\,. \quad (5.13)$$

Since however $\gamma = \tilde{\gamma} + (y-x)\eta + x$ runs for $\tilde{\gamma} \in \mathcal{H}_o$ through exactly all paths of finite kinetic energy $\frac{1}{2}\int_o^t |\frac{d\gamma}{d\tau}|^2 d\tau$ that start at y for $t = 0$ and end at x for $\tau = t$, and the kinetic energy is equal

$$\frac{1}{2}\int_o^t |\frac{d\gamma}{d\tau}|^2 d\tau = \frac{1}{2}|\eta|^2 (y-x)^2 + \int_o^t |\frac{d\tilde{\gamma}}{d\tau}|^2 d\tau\,, \quad (5.14)$$

we shall also write (5.13) shortly as

$$G_t^\hbar(x,y) = (2\pi i\hbar t)^{-\frac{d}{2}} \widetilde{\int_{\substack{\gamma(0)=y\\ \gamma(t)=x}}} e^{\frac{i}{2\hbar}\int_o^t |\frac{d\gamma}{d\tau}|^2 d\tau} e^{-\frac{i}{\hbar}\int_o^t V(\gamma(\tau))d\tau} d\gamma \quad (5.15)$$

or also

$$G_t^\hbar(x,y) = (2\pi i\hbar t)^{-\frac{d}{2}} \widetilde{\int_{\substack{\gamma(0)=y\\ \gamma(t)=x}}} e^{\frac{i}{\hbar}S} d\gamma\,. \quad (5.16)$$

We see that, for a fixed $x \in R^d$, $G_t(x,y)$ as a function of y is of the form (4.2) with $I(h,y) \sim G_t(x,y)$ and $h = \hbar$, and the Hilbert space $\mathcal{H} = \mathcal{H}_o$ of paths with finite kinetic energy such that $\gamma(0) = \gamma(t) = 0$. Let us now assume for the potential V that

$$V(x) = \int_{R^d} e^{ix\cdot\beta}\, d\nu(\beta) \tag{5.17}$$

with

$$\int e^{|\beta|\cdot\varepsilon}\, d|\nu|(\beta) < \infty \tag{5.18}$$

for some $\varepsilon > 0$. For $\gamma \in \mathcal{H}_0$ we set $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1(\frac{kt}{n}) = 0$, $k = 0,1,\ldots,n$ and γ_2 is piecewise linear with discontinuities of the derivatives only at the points $\frac{kt}{n}$, $k = 1,\ldots,n-1$. We see immediately that $\gamma = \gamma_1 + \gamma_2$ is a direct decomposition of $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\dim \mathcal{H}_2 = n\cdot d < \infty$. Moreover for any $\beta \in R^d$ and $s \in [0,t]$ the linear functional $\beta\cdot\gamma(s) \in \mathcal{H}_0^*$ decomposes in a direct sum

$$\beta\cdot\gamma(s) = \beta\cdot\gamma_1(s) + \beta\cdot\gamma_2(s) \tag{5.19}$$

with the following norm estimate for the first component

$$|\beta\cdot\gamma_1(s)| \leq |\beta|\sqrt{t/n}\cdot|\gamma_1| \ , \tag{5.20}$$

because if $\frac{kt}{n} \leq s < \frac{(k+1)t}{n}$ then

$$|\beta\cdot\gamma_1(s)| = \left|\int_{\frac{kt}{n}}^{s} \beta\cdot\frac{d\gamma_1}{d\tau}\, d\tau\right| \leq |\beta| \int_{\frac{kt}{n}}^{s} \left|\frac{d\gamma_1}{d\tau}\right| d\tau \leq |\beta|\sqrt{t/n}|\gamma_1| \ .$$

Now, with $\gamma = \gamma_1 + \gamma_2 \in \mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$, we consider

$$V(\gamma_1,\gamma_2,y) = \int_0^t V(\gamma(\tau) + (y-x)\eta(\tau) + x)\, d\tau \ , \tag{5.21}$$

where $\eta(\tau) = 1 - \frac{\tau}{t}$. Then we have

$$V(\gamma_1,\gamma_2,y) = \int_0^t \int_{R^d} e^{i\beta\gamma_1(\tau)}\, e^{i\beta\gamma_2(\tau)}\, e^{i((y-x)\eta(\tau)+x)\beta}\, d\nu(\beta)\, d\tau \ . \tag{5.22}$$

Hence by (5.20) we have that if

$$t\int_{R^d} e^{\sqrt{2}\lambda|\beta|\sqrt{t/n}} d|\nu|(\beta) < \lambda^2 \tag{5.23}$$

then $V(\gamma,y) = V(\gamma_1,\gamma_2,y)$ satisfies the conditions of theorem 4.1. By the condition (5.18) on the measure ν we see that there are, for any t, numbers $\lambda > 0$ and n such that (5.23) is satisfied. Hence we may apply theorem 4.1 to the function $G_t^{\hbar}(x,y)$ given in (5.16). So that the following function

$$\psi_{\hbar}(x,t) = \int G_t^{\hbar}(x,y)\, e^{\frac{i}{\hbar}f(y)} \chi(y)dy \tag{5.24}$$

is, in the notation of theorem 4.1, of the form $I(h,-f)$. Hence we may use theorem 4.1 to study the asymptotic behavior as $\hbar \to 0$. We remark that the results here are not new, but we like to state them anyhow as an example of the application of theorem 4.1.

First we compute the singular locus S_V and we see by (4.5) that $(\gamma,y) \in S_V$ iff $\bar{\gamma} = \gamma + (y-x)\eta + x$ is a classical path such that $\bar{\gamma}(t) = x$ and $\bar{\gamma}(0) = y$, i.e. $\bar{\gamma}$ satisfies the differential equation

$$m\frac{d^2}{d\tau^2}\bar{\gamma} = -\nabla V(\bar{\gamma}) . \tag{5.25}$$

Furthermore we get by (4.14) that the mapping $S_V \to \Lambda_V$ of the singular locus into the Lagrange manifold Λ_V is given by

$$(\gamma,y) \to (y, d_y\bar{S}(x,y)) , \tag{5.26}$$

where $\bar{S}(x,y)$ is the classical action (5.1) computed along the classical path (5.25). Now since

$$\bar{S}(x,y) = \int_0^t \left(\frac{m}{2}\left|\frac{d\bar{\gamma}}{d\tau}\right|^2 - V(\bar{\gamma}(\tau))\right)d\tau \tag{5.27}$$

we have

$$d_y\bar{S} = m\int_0^t \left(\frac{d\bar{\gamma}}{d\tau}\cdot\frac{d}{d\tau}(d_y\bar{\gamma}(\tau)) - \nabla V(\bar{\gamma})\cdot d_y\bar{\gamma}(\tau)\right)d\tau \ .$$

So that

$$d_y\bar{S} = -\int_0^t \left(m\frac{d^2\bar{\gamma}}{d\tau^2} + \nabla V(\bar{\gamma})\right)d_y\bar{\gamma}(\tau)d\tau - m\dot{\bar{\gamma}}(0)d_y\bar{\gamma}(0) \ .$$

From the fact that γ satisfies (5.25) and $\bar{\gamma}(0) = y$ we see that

$$d_y\bar{S}(x,y) = -m\dot{\bar{\gamma}}(0) \tag{5.28}$$

and also, from the calculation above, that

$$d_x\bar{S}(x,y) = m\dot{\bar{\gamma}}(t) \ . \tag{5.29}$$

So that Λ_V in this case just consists of the points (y,p) in the phase space T^*Y, where p is a momentum at y of a classical particle that starts at x and ends at y in the time interval $[0,t]$.

The condition on the phase function $f(y)$ is then that the Lagrange manifold $(y,-\nabla f(y))$ intersects Λ_V transversally and the contribution to the limit as $\hbar \to 0$ in theorem 4.1 comes therefore from all classical paths that start at y with momentum $\nabla f(y)$ and end at x. We can now formulate these results in the following

<u>Theorem 5.1.</u> Consider the Schrödinger equation in R^d

$$i\hbar\frac{\partial}{\partial t}\psi_\hbar(x,t) = \left(-\frac{\hbar^2}{2}\Delta + V(x)\right)\psi_\hbar(x,t),$$

where the potential is the Fouriertransform of some complex measure ν such that

$$V(x) = \int_{R^d} e^{ix\beta}\, d\nu(\beta)$$

with

$$\int e^{|\beta|\epsilon}\, d|\nu|(\beta) < \infty$$

for some $\epsilon > 0$.

Let the initial condition be

$$\psi_\hbar\ (y,0) = e^{\frac{i}{\hbar}f(y)}\ \chi(y)$$

with $\chi \in C_0^\infty(R^d)$ and $f \in C^\infty(R^d)$ and such that the Lagrange manifold $L_f \equiv (y, -\nabla f)$ intersects the subset Λ_V of phase space transversally, where Λ_V is the subset of all points (y,p) of phase space such that p is momentum at y of a classical particle that starts at time zero from x, moves under the action of V and ends at y at time t.

Then the conclusions of Theorem 4.1 hold, with

$$\psi_\hbar\ (x,t) = I(\hbar, -f)\ .$$

In particular $\psi_\hbar\ (x,t)$ has an asymptotic expansion in powers of h, whose leading term is the sum of the values of the function

$$|Vol_V|^{\frac{1}{2}}\ e^{-\frac{1}{2}\pi i m}\ e^{\frac{i}{\hbar}S}\ e^{\frac{i}{\hbar}f}\ \chi$$

taken at the points $y^{(j)}$ such that a classical particle starting from $y^{(j)}$ at time zero with momentum $\nabla f(y^{(j)})$ is in x at time t. S is the classical action along this classical path $\bar{\gamma}^{(j)}$ and $m = -[n(1 - d_{11}V) + n(A_{-f})]$, in the notations of Theorem 4.1, is the Maslov index of the path $\bar{\gamma}(j)$.

Let $\bar{\gamma}_k^{(j)}$ and $y_l^{(j)}$ be the k-th respectively l-th component of $\bar{\gamma}^{(j)}$ respectively $y^{(j)}$, then we have

$$|Vol_V|^{\frac{1}{2}} = |\det\ ((\frac{\partial\bar{\gamma}_k(j)}{\partial y_l(j)}\ (y^{(j)},\ t)))|^{-\frac{1}{2}}\ .$$

Remark: It is clear that, at this point, we could continue the discussion of the approach to the classical limit in all details, following essentially the same lines as e.g. in Maslov, particularly [2], 1). However we shall not do this here, our aim in this section having been only to provide an example in which the general theory of oscillatory integrals in infinitely many dimensions and their asymptotic expansions can be applied, through Feynman path integrals as defined in [10]. Since in fact, as we have seen using the theory of the previous section, the whole discussion is reduced to the study of oscillatory integrals in finitely many dimensions, for which a well developed theory is available, one recovers in this way the asymptotic series in powers of $\hbar$ for the solution of Schrödinger's equation, hence a detailed analysis of the asymptotic approach to the classical limit of quantum mechanics.

Footnote

1) Concerning the general problem of the relations between quantum mechanics and classical mechanics there exist many other app approaches e.g. [16]. An approach centered around the quantization conditions of the "old quantum theory" have found recently increasing interest, and is also part of Maslov's theory as well as of the global theory of Fourier integral operators, see e.g. [17]. An early particularly striking example of such connections between classical and quantum mechanical quantities follows from the work of A. Huber [18], as pointed out by Fierz [19] to one of us about twelve years ago.

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